Radical Functions & Graphing Part 1

Key

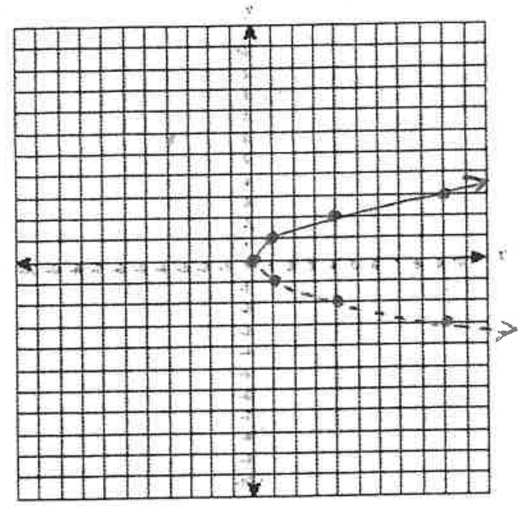
Graph $y = \sqrt{x}$ using a table of values and state the domain and range.

x	y
-4	undef
-1	undef
0	0
1	1
4	2
9	3

$y = \sqrt{-4}$ undef

$y = \sqrt{0} = 0$

$y = \sqrt{1} = 1$



Domain: $x \geq 0$

Range: $y \geq 0$

Notice the domain is restricted. Why? What is an easy way to obtain the domain?

The domain is restricted because an even index radicand cannot be negative

Domain can be found by setting radicand ≥ 0 and solving for variable

Radical functions can be graphed using transformations – just like quadratic functions! For example, $y = a\sqrt{x-h} + k$. The vertex is (h, k), and the vertical expansion/compress is a.

When graphing parabolas, what happened to $y = x^2$ when it became $y = -x^2$ (ie. When the a value was -1)? the parabola reflected through the horizontal line through the vertex

This is called a vertical reflection (a reflection through the horiz line through vertex).

Therefore, graph $y = -\sqrt{x}$ without a table of values on the grid above, and then state the domain and range below: D: $x \geq 0$ R: $y \leq 0$

What is the 'basic count' for a radical graph? Right 1 Up 1

from vertex: R 4 Up 2

Remember, the a value alters the up/down count. R 9 Up 3

etc.

> To find the x-intercept, set y = 0, and solve.
> To find the y-intercept, set x = 0, and solve.

Graph $y = 3\sqrt{x}$ and $y = -\frac{1}{2}\sqrt{x}$ on the graph below & state the *domain, range, x-int, y-int*

① $y = 3\sqrt{x}$

multiply 'up' counts by 3

vertex (0, 0)

right 1, up 3

4 6

9 9

D: $x \geq 0$ | x-int: 0

R: $y \geq 0$ | y-int: 0

② $y = -\frac{1}{2}\sqrt{x}$

multiply 'down' counts by $\frac{1}{2}$. vertex (0, 0)

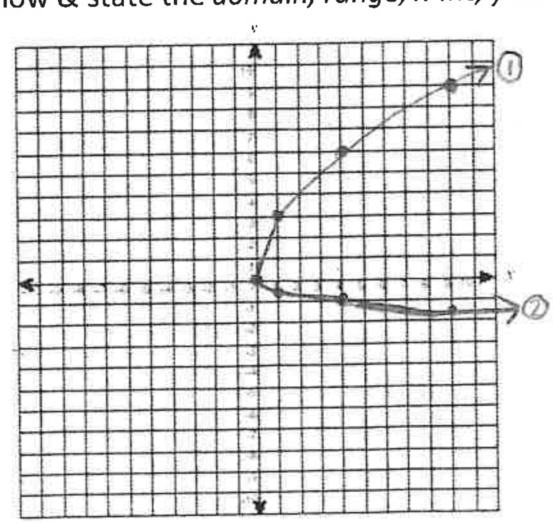
right 1, down $\frac{1}{2}$

4 down 1

9 down 1.5

D: $x \geq 0$ x-int: 0

R: $y \leq 0$ y-int: 0



Graph $y = \sqrt{x+3} + 2$ and $y = -2\sqrt{x-1} - 4$. Give *D, R, x-int, y-int* for each.

① $y = \sqrt{x+3} + 2$

vertex $(-3, 2)$

right	up
1	1
4	2
9	3

D: $x \geq -3$

R: $y \geq 2$

x-int: $\emptyset$

y-int: $2 + \sqrt{3}$

$y = \sqrt{0+3} + 2$

② $y = -2\sqrt{x-1} - 4$

vertex $(1, -4)$

$a = -2$

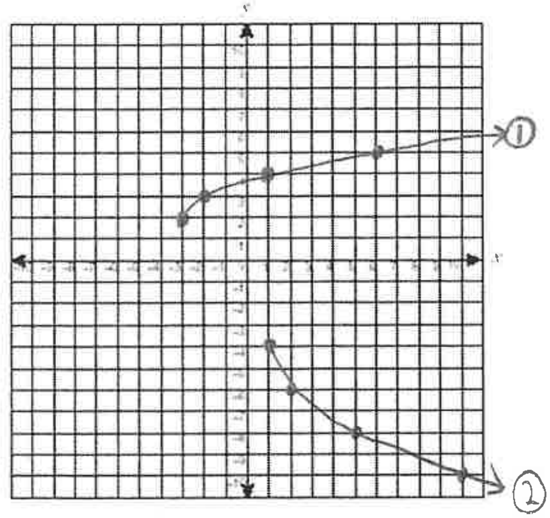
right	down
1	2
4	4
9	6

D: $x \geq 1$

R: $y \leq -4$

x-int: $\emptyset$

y-int: $\emptyset$



So, using *transformation theory* (using *a, h, k*)that was learned for quadratics, we can graph other types of functions with the same principles.

The graph $y = -\sqrt{x}$ is a reflection through the *horizontal line through the vertex* of the $y = \sqrt{x}$ graph. Using a table of values, graph $y = \sqrt{-x}$ below, and state the domain and range.

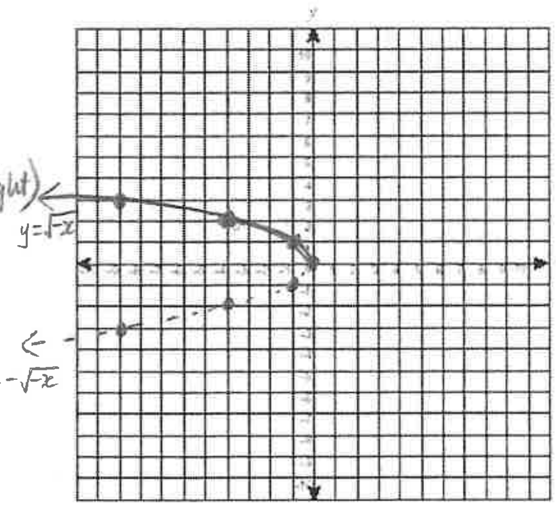
$y = \sqrt{-(-9)} = \sqrt{9} = 3$

x	y
-9	3
-4	2
-1	1
0	0
1	$\emptyset$

$y = \sqrt{-x}$

left (instead of right)

left	up
1	1
4	2
9	3



D: $x \leq 0$

R: $y \geq 0$

D: $-x \geq 0$

$\div -1$ flip $\div -1$

$x \leq 0$

On the grid above, graph $y = -\sqrt{-x}$ without a table of values, and state the domain and range.

down left vertex $(0,0)$

left	down
1	1
4	2
9	3

For the following radical functions, state the *domain, range, x-int,* and *y-int.*

a) $y = -4\sqrt{x-3} + 8$

D: $x \geq 3$

R: $y \leq 8$

y-int: $\emptyset$

x-int: 7

D: $x - 3 \geq 0$

$x \geq 3$

R: vertex $(3, 8)$

- opens DOWN; $a = -4$

y-int. $y = -4\sqrt{0-3} + 8$

$= -4\sqrt{-3} + 8$

undef

x-int:

$0 = -4\sqrt{x-3} + 8$

$-8 = -4\sqrt{x-3}$

$2 = \sqrt{x-3}$

$4 = x - 3$

$x = 7$

b) $y = \frac{1}{2}\sqrt{x+5} - 1$

D: $x \geq -5$

R: $y \geq -1$

y-int: $-1 + \frac{\sqrt{5}}{2}$

x-int: -1

x-int

$0 = \frac{1}{2}\sqrt{x+5} - 1$

$1 = \frac{1}{2}\sqrt{x+5}$

$2 = \sqrt{x+5}$

$4 = x + 5$

$x = -1$

D: $x + 5 \geq 0$

$x \geq -5$

R: vertex $(-5, -1)$

$a = \frac{1}{2}$ so opens up.

y-int $y = \frac{1}{2}\sqrt{0+5} - 1$

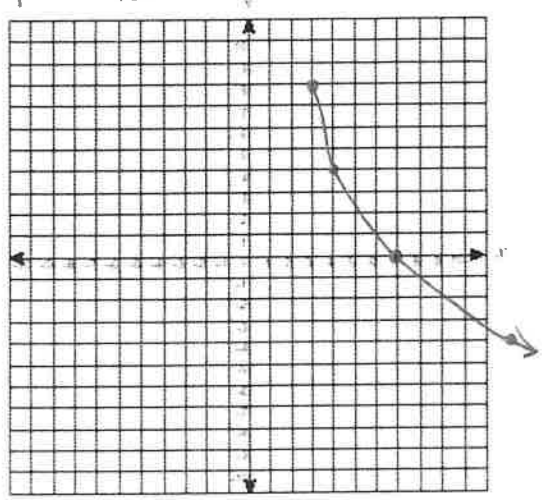
$y = -1 + \frac{\sqrt{5}}{2}$

Graph the following functions. A good way to do this is to locate the vertex first, and then establish the 'count' from the vertex by altering the 'basic count' accordingly.

a) $y = -4\sqrt{x-3} + 8$

vertex (3,8)

right 1, down 4
4 8
9 12



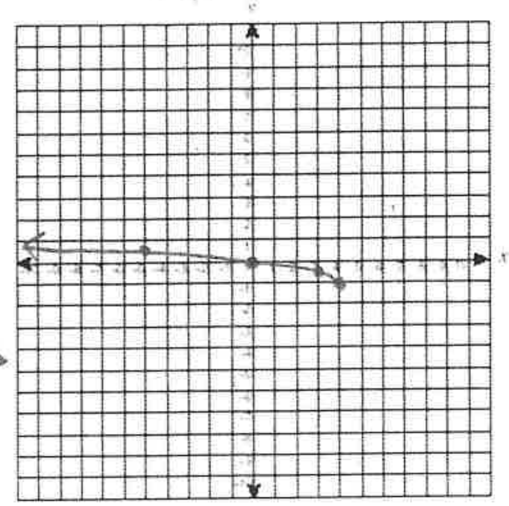
b) $y = \frac{1}{2}\sqrt{4-x} - 1$

$y = \frac{1}{2}\sqrt{-(x-4)} - 1$

count left!

vertex (4,−1)

left 1, up $\frac{1}{2}$
4 up 1
9 $1\frac{1}{2}$



Here is a summary of transformations for radical graphs:

$$y = \pm a\sqrt{\pm(x-h)} + k$$

count up (+) down (−)

multiply up/down counts by 'a'

count right (+) left (−)

vertex (h, k)
h = horiz trans
k = vert trans

Basic Count:	right 1, up 1
	right 4, up 2
	right 9, up 3

<u>**Steps for graphing:**</u>

1) **If x is negative, factor it out.**
2) Locate the vertex (h, k)
3) From the vertex, build the count:
   If no negative in front **in** radical, go right. If negative in front in radical, go left.
   If ***a*** is positive, count up. If ***a*** is negative, count down. Multiply up/down count by ***a*** value.

Graph the following and state the *domain, range, x-intercept*, and *y-intercept*

$y - 6 = -2\sqrt{3-x}$

$y = -2\sqrt{-(x-3)} + 6$

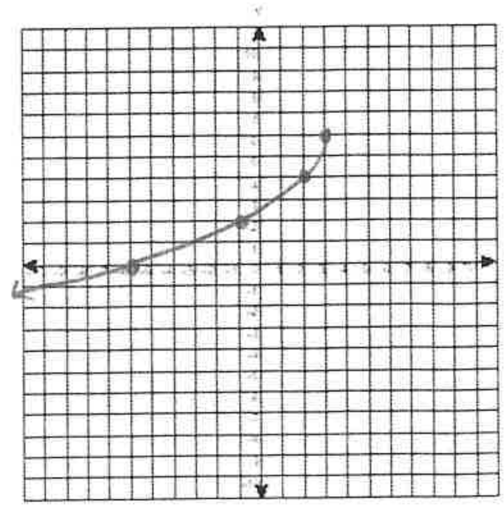
count down

count left

vertex (3,6)

left 1, down 2
4 4
9 6

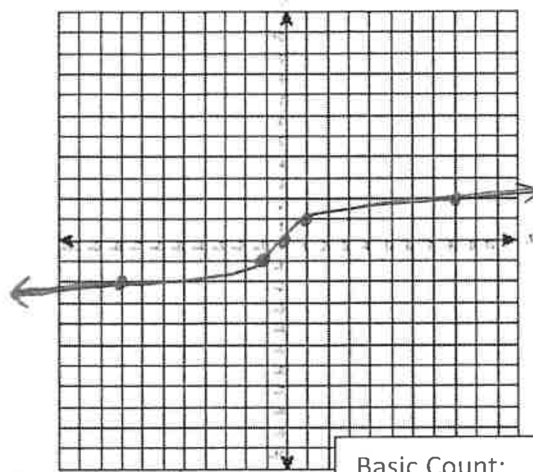
D: $x \leq 3$
R: $y \leq 6$
x-int: −6
y-int: $6 - 2\sqrt{3}$



Radical Functions & Graphing Part 2

We've learned how to graph radical functions with index *2* using transformations. How do you graph $y = \sqrt[3]{x}$? Use a table of values:

x	y
8	2
1	1
0	0
-1	-1
-8	-2



D: $x \in \mathbb{R}$

R: $y \in \mathbb{R}$

Basic Count:

right 1 up 1
8 2

left 1 down 1
8 2

Notice the domain is not restricted. Why?

It's a cube root (odd index) so radicand can be positive, zero, or negative.

When is the domain of a radical function restricted, and when is it not?

odd index ⇒ no restrictions

even index ⇒ radicand must be non-negative

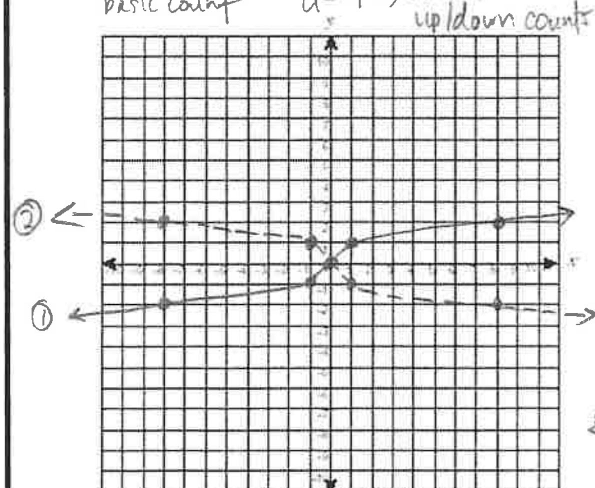
On the grids below, sketch 1) $y = \sqrt[3]{x}$ (2) $y = -\sqrt[3]{x}$ (3) $y = \sqrt[3]{-x}$ (4) $y = -\sqrt[3]{-x} - 5$

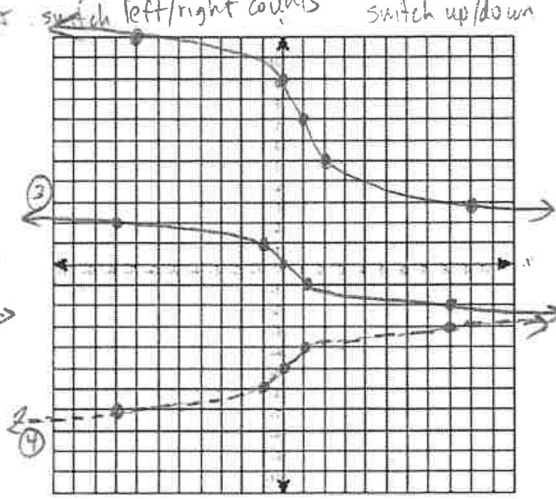
1) $y = \sqrt[3]{x}$
vertex (0,0)
basic count

2) $y = -\sqrt[3]{x}$
vertex (0,0)
$a = -1$ ⇒ switch up/down counts

③ $y = \sqrt[3]{-x}$
vertex (0,0)
switch left/right counts

④ $y = -\sqrt[3]{-x} - 5$
vertex (0,-5)
switch up/down and switch left/right





Are the domains or ranges of any of the graphs restricted in any way? no!

On the second grid above, graph $y = -2\sqrt[3]{x-1} + 7$. State the *D, R, x-int, y-int*:

vertex (1,7)

$a = -2$

switch up/down and double counts.

		basic	now
right	1	up 1	down 2
	8	up 2	down 4
left	1	down 1	up 2
	8	down 2	up 4

D: $x \in \mathbb{R}$

R: $y \in \mathbb{R}$

y-int : 9

x-int $\frac{351}{8}$

x-int

$0 = -2\sqrt[3]{x-1} + 7$

$-7 = -2\sqrt[3]{x-1}$

$\frac{7}{2} = \sqrt[3]{x-1}$

$\frac{343}{8} = x - 1$

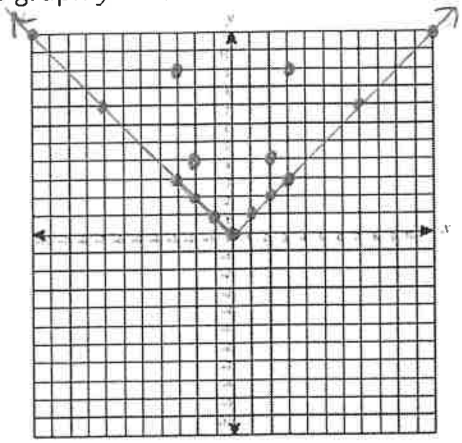
$x = \frac{351}{8}$

Graphing $y = \sqrt{f(x)}$

To graph $y = \sqrt{x^2}$, first graph $y = x^2$ and use it to help graph $y = \sqrt{x^2}$. State the domain and range of $y = \sqrt{x^2}$

$D: x \in \mathbb{R}$

$R: y \geq 0$



Once you have the graph of $y = x^2$, simply square root each y value to get the y values for $y = \sqrt{x^2}$

Where $y = 9$ on original (at $x = \pm 3$),

on $y = \sqrt{x^2}$, it is $\sqrt{9} = 3$

$\sqrt{4} = 2$

$\sqrt{1} = 1$

$\sqrt{0} = 0$

Find a few more values $x = 7$, $x = 11$

$y = \sqrt{7^2} = \sqrt{49} = 7$ so $(7, 7)$, also $(-7, 7)$

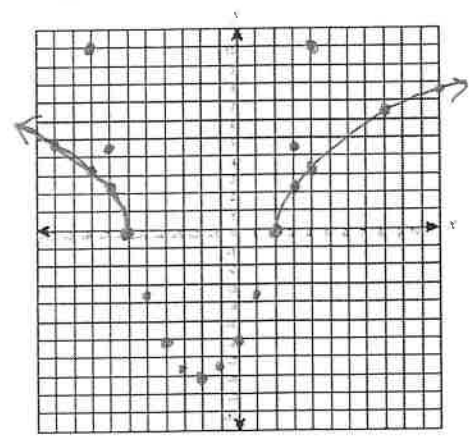
$y = \sqrt{11^2} = \sqrt{121} = 11$ so $(11, 11)$, also $(-11, 11)$

Graph $y = \sqrt{\frac{1}{2}(x+2)^2 - 8}$ and state the domain and range.

Graph $y = \frac{1}{2}(x+2)^2 - 8$ first

vertex $(-2, -8)$

$a = \frac{1}{2}$



when $x = -8, 4$ $\quad y = \sqrt{10} \cong 3.2$

when $x = -7, 3$ $\quad y = \sqrt{4.5} \cong 2.1$

when $x = -6, 2$ $\quad y = \sqrt{0} = 0$

when $x = -5, 1$ $\quad y = \sqrt{-3.5} = \emptyset$

etc.

$D: x \leq -6,\ x \geq 2$

$R: y \geq 0$

when $x = -10$, $y = \sqrt{\frac{1}{2}(-10+2)^2 - 8} = \sqrt{24} \cong 4.9$

when $x = 8$, $y = \sqrt{\frac{1}{2}(8+2)^2 - 8} = \sqrt{42} \cong 6.5$

$x = 11$, $y = \sqrt{\frac{1}{2}(11+2)^2 - 8} = \sqrt{76.5} \cong 8.7$

In general, when graphing the square root of a function $y = \sqrt{f(x)}$:

1) Graph the function without the square root $y = f(x)$

2) Square root each of the y values of the original function to get the y values for the square root function. Any negative y values from $y = f(x)$ will not exist in $y = \sqrt{f(x)}$

3) Get any extra points by choosing applicable x values and solving for y in $y = \sqrt{f(x)}$

Absolute Value Equations

What is the **absolute value** of a number? The distance the number is from zero regardless of direction

Example 1 – Simplify a) $|6|$ b) $|-4|$ c) $\left|-\frac{7}{3}\right|$

$= 6$ $= 4$ $= \frac{7}{3}$

Example 2 – Solve $|x| = 2$

$x = 2$ and $x = -2$ (both have a distance of 2 units from zero)

Steps for solving an absolute value equation:

1) Get the absolute value by itself on one side (everything not in the absolute value should be on the other side).
2) Set up two cases: the positive case and the negative case. Solve for each case.
3) Check each solution to see if it is an actual or *extraneous* solution.

Example 3 – Solve $|x + 7| = 10$ First, by inspection: how can we make $|x+7|$ either 10 or -10

$x = 3, -17$

Positive Case:	Negative Case:	Check: $x=3$	$x=-17$
$x+7=10$ $x=3$	$-(x+7)=10$ $x+7=-10$ $x=-17$	LS | RS $\lvert 3+7\rvert$ | 10 $\lvert 10\rvert$ 10 ✓	LS | RS $\lvert x+7\rvert$ | 10 $\lvert -17+7\rvert$ $\lvert -10\rvert$ 10 ✓

Example 4 – Solve $|x - 2| + 3 = 9 \Rightarrow |x-2| = 6$

Positive Case:	Negative Case:	Check: $x=8$	$x=-4$
$x-2=6$ $x=8$	$x-2=-6$ $x=-4$	LS | RS $\lvert 8-2\rvert+3$ | 9 $\lvert 6\rvert+3$ $6+3$ 9 ✓	LS | RS $\lvert -4-2\rvert+3$ | 9 $\lvert -6\rvert+3$ $6+3$ 9 ✓

Example 5 – Solve $|3x - 2| = 1 - x$ algebraically

Pos	Neg
$3x-2=1-x$	$3x-2=-(1-x)$
$4x=3$	$3x-2=-1+x$
$x=\frac{3}{4}$	$2x=1$
	$x=\frac{1}{2}$

Check: $x=\frac{3}{4}$

LS	RS
$\lvert 3(\frac{3}{4})-2\rvert$	$1-\frac{3}{4}$
$\lvert \frac{9}{4}-2\rvert$	$\frac{1}{4}$
$\lvert \frac{1}{4}\rvert$	
$\frac{1}{4}$	✓

$x=\frac{1}{2}$

LS	RS
$\lvert 3(\frac{1}{2})-2\rvert$	$1-\frac{1}{2}$
$\lvert -\frac{1}{2}\rvert$	$\frac{1}{2}$
$\frac{1}{2}$	✓

Example 6 – Solve $|x - 3| + 7 = 4$

$-7 \quad -7$

$|x-3|=-3$

Pos	Neg
$x-3=-3$	$x-3=-(-3)$
$x=0$	$x-3=3$
extraneous	$x=6$
	extraneous

NO SOLUTIONS ($\varnothing$)

Check: $x=0$

LS	RS
$\lvert 0-3\rvert+7$	4
$\lvert -3\rvert+7$	
$3+7$	
10	✗

$x=6$

LS	RS
$\lvert 6-3\rvert+7$	4
$\lvert 3\rvert+7$	
$3+7$	
10	✗

no solutions

<u>An Absolute Value Equation with No Solution:</u> If there is just an absolute value on one side, and just a negative constant on the other, there are no solutions!

ex $|x-4|=-5$ is NO SOLUTIONS

Example 7 – Solve $|4x - 5| + 2 = 2 \Rightarrow |4x-5|=0$

$-2 \quad -2$

* Positive + negative case are the same as negative zero is just zero, so only one solution

$4x-5=0$

$4x=5$

$x=\frac{5}{4}$

LS	RS
$\lvert 4(\frac{5}{4})-5\rvert+2$	2
$\lvert 5-5\rvert+2$	
$\lvert 0\rvert+2$	
$0+2$	✓
2	

Example 8 – Solve $|x + 5| = 4x - 1$ algebraicially

Pos

$x+5=4x-1$

$-3x=-6$

$x=2$

Neg

$x+5=-(4x-1)$

$x+5=-4x+1$

$5x=-4$

$x=\frac{-4}{5}$

↑ extraneous

Check: $x=2$

LS	RS
$\|2+5\|$	$4(2)-1$
$\|7\|$	$8-1$
7	7 ✓

$x=-\frac{4}{5}$

LS	RS
$\left\|\frac{-4}{5}+5\right\|$	$4\left(\frac{-4}{5}\right)-1$
$\left\|\frac{-4}{5}+\frac{25}{5}\right\|$	$\frac{-16}{5}-\frac{5}{5}$
$\left\|\frac{21}{5}\right\|$	$\frac{-21}{5}$
$\frac{21}{5}$	✗

quadratic absolute value equations

Example 9 – Solve $|x^2 - 7x + 2| = 10$

Pos

$x^2-7x+2=10$

$x^2-7x-8=0$

$(x-8)(x+1)=0$

$x=-1,8$

Neg

$x^2-7x+2=-10$

$x^2-7x+12=0$

$(x-4)(x-3)=0$

$x=3,4$

Check: $x=-1$

LS	RS
$\|(-1)^2-7(-1)+2\|$	10
$\|1+7+2\|$	
$\|10\|$	
10	✓

$x=3$

LS	RS
$\|3^2-7(3)+2\|$	10
$\|9-21+2\|$	
$\|-10\|$	
10	✓

$x=8$

LS	RS
$\|(8)^2-7(8)+2\|$	10
$\|64-56+2\|$	
$\|10\|$	
10	✓

$x=4$

LS	RS
$\|4^2-7(4)+2\|$	10
$\|16-28+2\|$	
$\|-10\|$	
10	✓

Example 10 – Solve $|x - 2| = x - 2$

right side must be 0 or positive to have a solution. Thus, solutions are any x values that cause the right side to be zero or positive, so:

$x-2\geq 0$

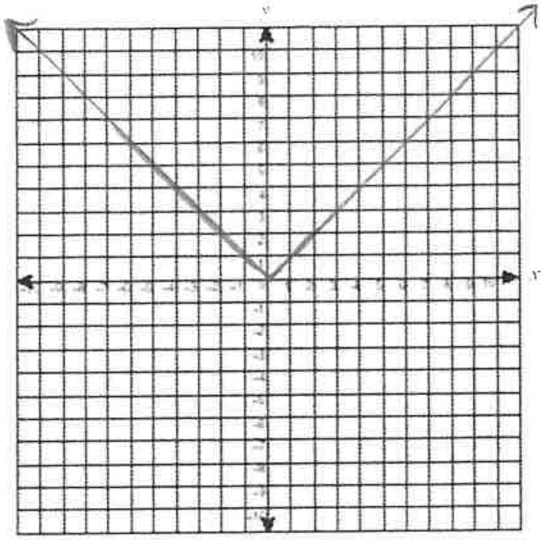
$x\geq 2$

Absolute Value Functions & Graphing

Example 1 – Graph $y = |x|$

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

D: $x \in \mathbb{R}$
R: $y \geq 0$



The graph $y = |x|$ consists of two graphs:

$y = x$ when $x \geq 0$
$y = -x$ when $x < 0$

This is why an absolute value graph is called a **piecewise function.**

At times, absolute value functions are set up analogous to quadratic functions in standard form, using ***h, k,*** and ***a*** values to graph the function.

What is the ***basic count***? over 1, up 1
2, 2
3, 3
etc.

Summarize how each component of the function affects the graph:

$y = \pm a|x - h| + k$ & $y = \pm a|-(x - h)| + k$

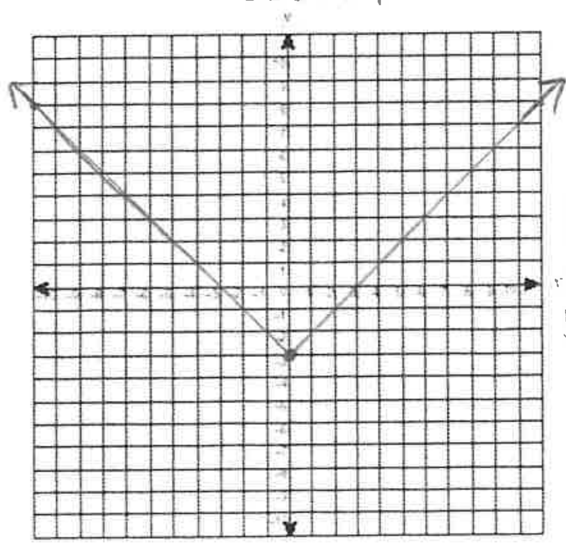
opens up (+) down (−)   multiply up/down count by 'a'   vertex (h, k)

causes a horizontal reflection in the vertical line through the vertex ⇒ for abs value this does not change the result!

Example 1 – Graph each function and state the *D, R, x-int, y-int*:

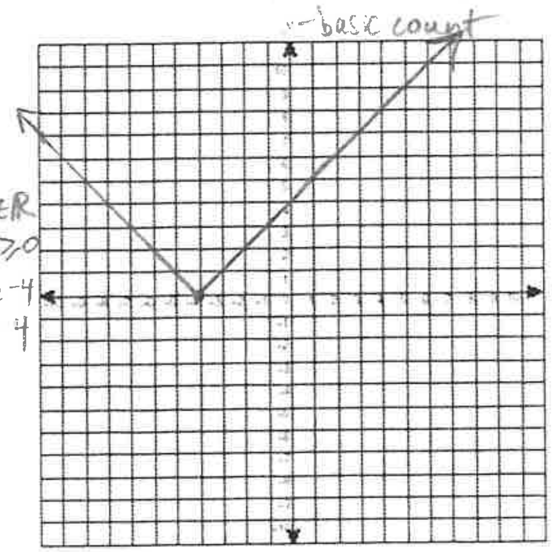
a) $y = |x| - 3$ – vertex $(0, -3)$ – basic count

D: $x \in \mathbb{R}$
R: $y \geq -3$
x-ints: −3, 3
y-int: 0



b) $y = |x + 4|$ – vertex $(-4, 0)$ – basic count

D: $x \in \mathbb{R}$
R: $y \geq 0$
x-int: −4
y-int: 4

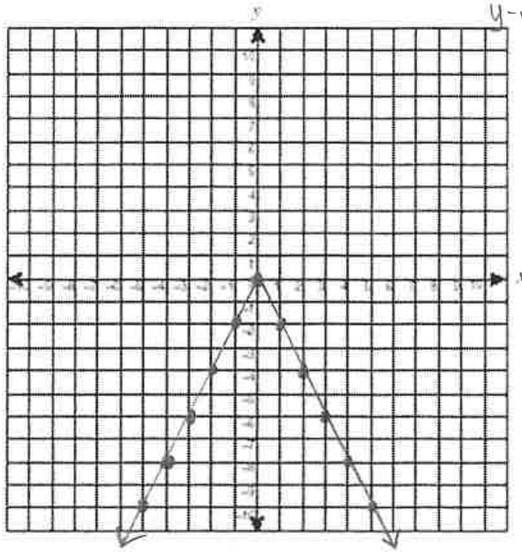


vertex $(0,0)$

over 1 down 2

2 4

c) $y = -2|x|$

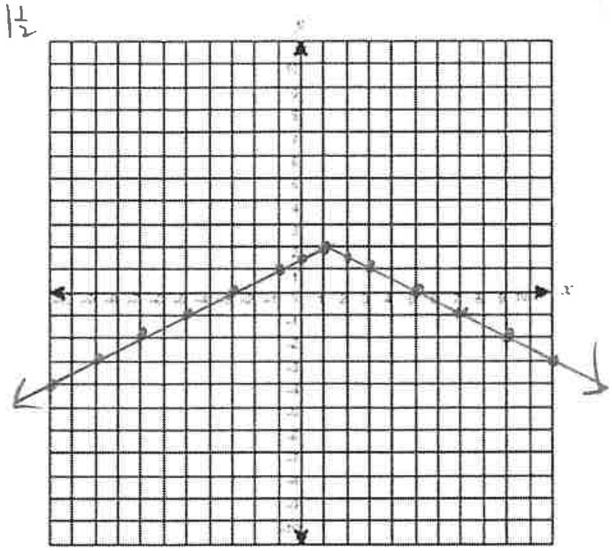


D: $x \in \mathbb{R}$
R: $y \le 0$
x-int $= 0$
y-int $= 0$

D: $x \in \mathbb{R}$
R: $y \le 2$
x-ints: $-3, 5$
y-int: $1\frac{1}{2}$

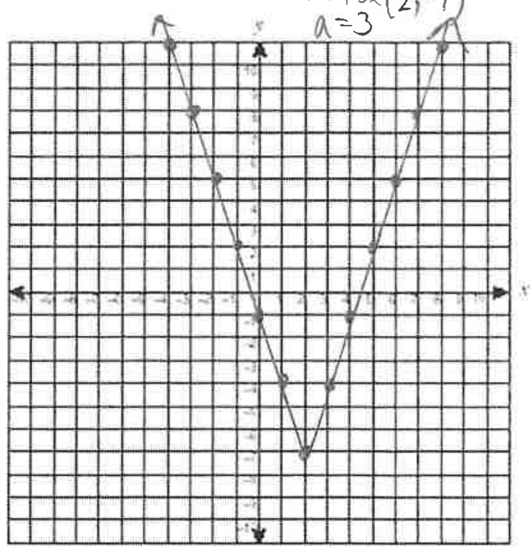
$y = -\frac{1}{2}|-(x-1)| + 2$ Vertex $(1,2)$

d) $y = -\frac{1}{2}|1 - x| + 2$ $a = -\frac{1}{2}$



$y = 3|-(x-2)| - 7$

e) $y + 7 = 3|2 - x|$ vertex $(2,-7)$ $a = 3$

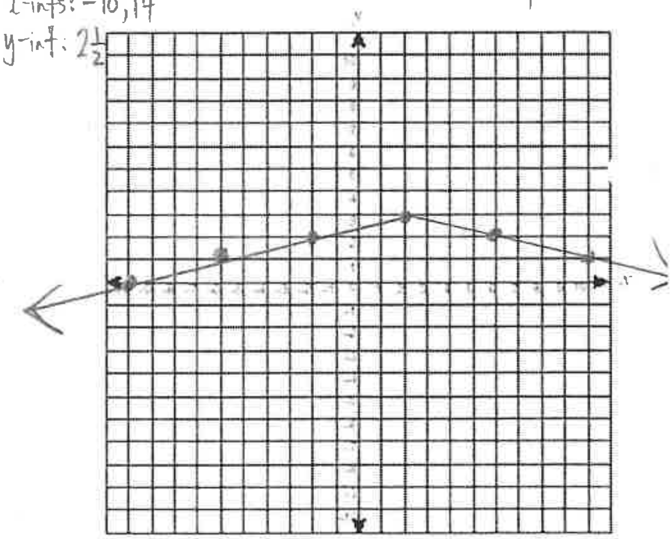


D: $x \in \mathbb{R}$
R: $y \ge -7$
x-ints: $-\frac{1}{3}, 4\frac{1}{3}$
y-int: -1

D: $x \in \mathbb{R}$
R: $y \le 3$
x-ints: $-10, 14$
y-int: $2\frac{1}{2}$

$y = -\frac{1}{4}|x-2| + 3$ vertex $(2,3)$

f) $y = 3 - \frac{1}{4}|x - 2|$ $a = -\frac{1}{4}$



Look at (c) above. The right branch only would have the equation $y = -2x + 0$ and the left branch only would have the equation $y = 2x + 0$. Thus we can write $y = -2|x|$ as a **piecewise function**:

$$y = \begin{cases} -2x & \text{when } x \ge 0 \\ 2x & \text{when } x < 0 \end{cases}$$

Write examples *d, e, & f* as piecewise functions:

(d)
$$y = \begin{cases} -\frac{1}{2}x + \frac{5}{2} & \text{when } x \ge 1 \\ \frac{1}{2}x + \frac{3}{2} & \text{when } x < 1 \end{cases}$$

(e)
$$y = \begin{cases} 3x - 13 & \text{when } x \ge 2 \\ -3x - 1 & \text{when } x < 2 \end{cases}$$

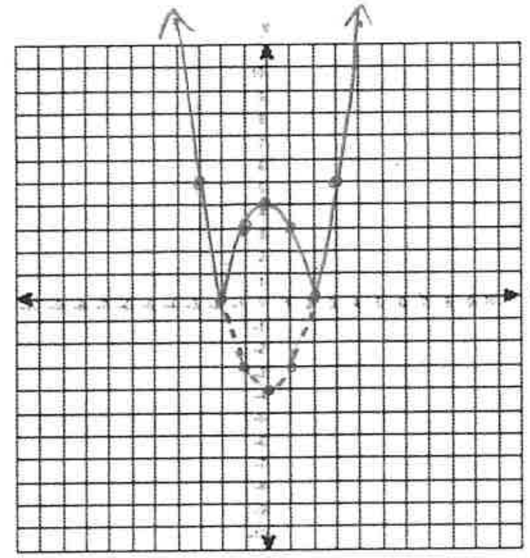
(f)
$$y = \begin{cases} -\frac{1}{4}x + \frac{7}{2} & \text{when } x \ge 2 \\ \frac{1}{4}x + \frac{5}{2} & \text{when } x < 2 \end{cases}$$

If a graph is constructed for any function $f(x)$, what will the graph for $|f(x)|$ look like?

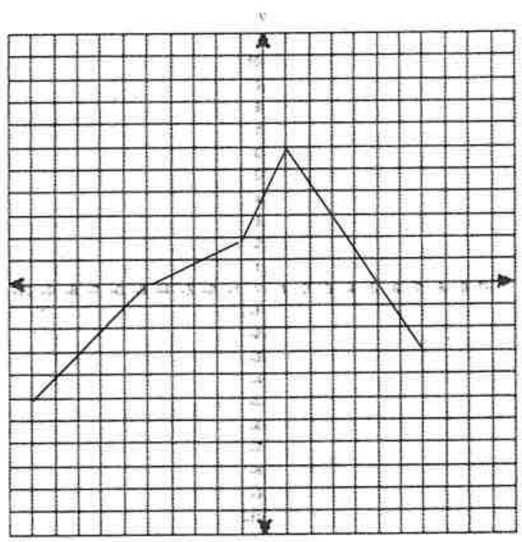
Example 3 – Graph $y = x^2 - 4$ using a table of values. Then graph $y = |x^2 - 4|$

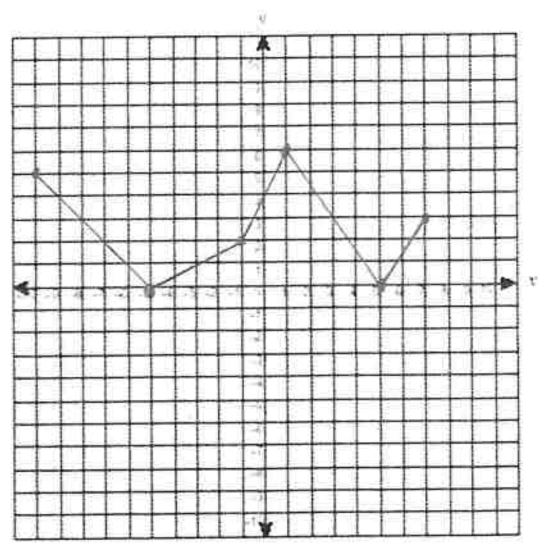
x	$f(x)$
-4	12
-3	5
-2	0
-1	-3
0	-4
1	-3
2	0
3	5
4	12

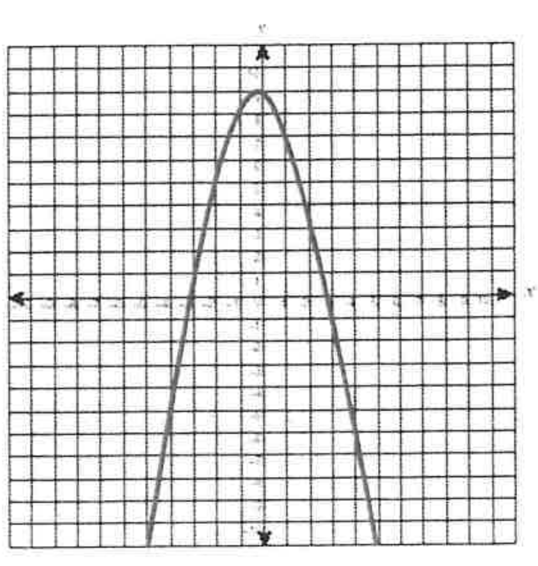
x	$\|f(x)\|$
-4	12
-3	5
-2	0
-1	3
0	4
1	3
2	0
3	5
4	12

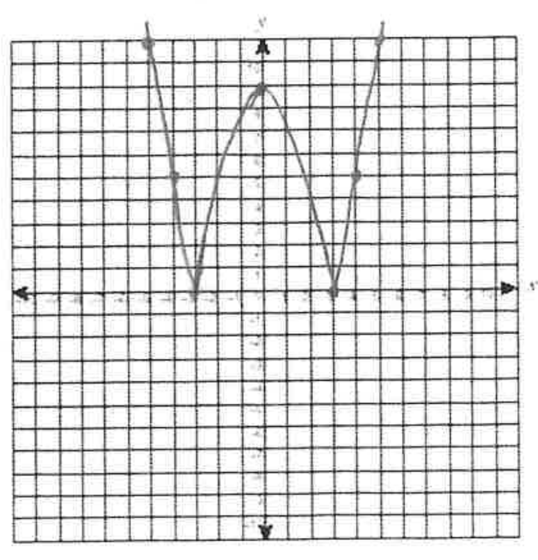


Example 4 – Given each graph of $y = f(x)$, graph $y = |f(x)|$.









4.5A – Rational Functions Part 1

A function f is a **rational function** if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials.

The domain of f consists of all real numbers except x values that make the denominator equal to zero (undefined values).

Example 1 – What are the undefined values for each rational function?

a) $\frac{2}{x+5}$ $x \neq -5$

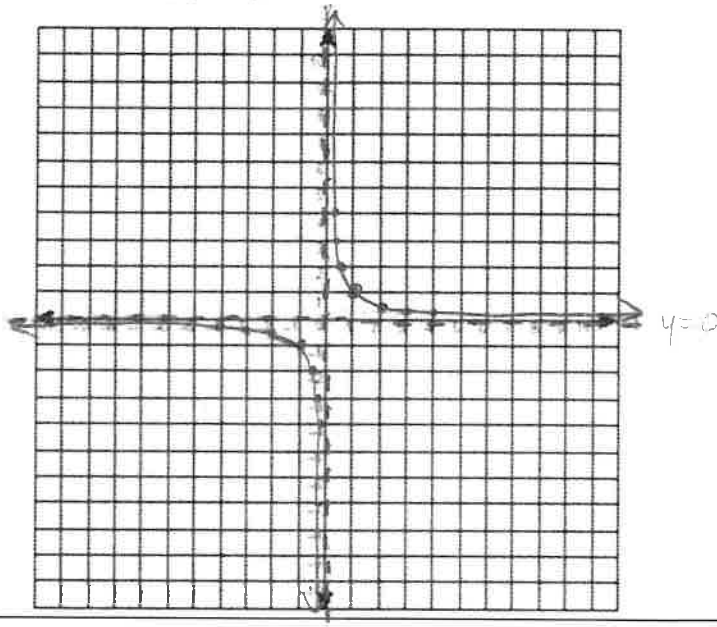
b) $\frac{x}{x^2-4} = \frac{x}{(x+2)(x-2)}$ $x \neq \pm 2$

c) $\frac{x+4}{x^2-2x-15}$ $(x-5)(x+3)$ $x \neq -3, 5$

d) $\frac{x-7}{x^2+16}$ None

Graphing Rational Functions

Example 2 – Graph $y = \frac{1}{x}$



x	y
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
10	$\frac{1}{10}$
100	$\frac{1}{100}$
1000	$\frac{1}{1000}$
0	undef
0.5	2
0.2	5
0.1	10
0.01	100
0.001	1000
-1	-1
-2	$-\frac{1}{2}$
-10	$-\frac{1}{10}$
-100	$-\frac{1}{100}$
-0.5	-2
-0.1	-10
-0.01	-100
-0.001	-1000

D: $x \neq 0$ R: $y \neq 0$

The function is not defined for $x = 0$, so this is a **vertical asymptote** of the function, and is drawn as a dashed line.

The pattern in the table can be written as:

as $x \to 0^+, f(x) \to +\infty$

as $x \to 0^-, f(x) \to -\infty$

as $x \to +\infty, f(x) \to 0^+$

as $x \to -\infty, f(x) \to 0^-$

The line defined by $y = 0$ is said to be a **horizontal asymptote** of the function, and is drawn as a dashed line.

An **asymptote** is not part of the graph. A vertical asymptote is a line the graph approaches as the denominator approaches zero. A horizontal asymptote Is a line the graph approaches as $|x|$ gets larger. Not every rational function has both a horizontal AND vertical asymptote.

Vertical Asymptotes are found when the denominator equals zero. Sometimes the denominator must be factored to find the vertical asymptotes (see Example 1 above).

Example 3 – Graph $y = \frac{1}{x-1} - 1$ $\boxed{x \neq 1}$

What is the vertical asymptote? Plot it.

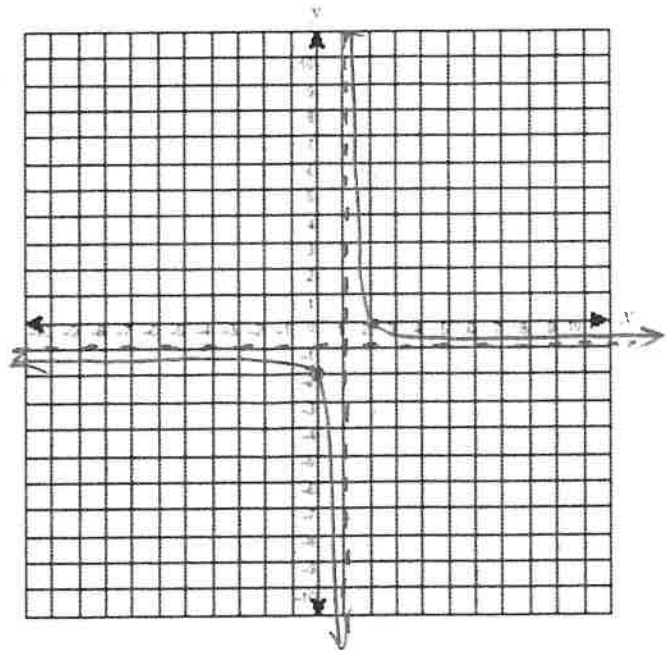
$x = 1$

Now, complete the table of values below:

$\frac{1}{0.999-1} - 1 = -\infty$

x	y
2	0
0	-2
1^-	$-\infty$
1^+	$+\infty$
-1000	$(-1)^-$
1000	$(-1)^+$

$\frac{1}{-1000-1} - 1 = (-1)^-$



D: $x \neq 1$ *R:* $y \neq -1$

x-int: 2 *y-int:* -2

Another way to think about the function above is using $y = \frac{1}{x-h} + k$, where the intersection of the two asymptotes is (h, k). So, $y = \frac{1}{x-1} - 1$ is just like $= \frac{1}{x}$, but shifted one unit right and one unit down. intersection of asymptotes is $(1, -1)$, then same shape as $y = \frac{1}{x}$

Before we graphed $y = \frac{1}{x-1} - 1$ above, we knew the vertical asymptote, but the horizontal asymptote wasn't apparent until after. How can we find the horizontal asymptote before?

The horizontal asymptote is the value that y approaches as x approaches $\pm\infty$. Consider the function $(x) = \frac{g(x)}{h(x)}$:

1) If $h(x)$ is a higher power than $g(x)$, the horizontal asymptote is $y = k$.
2) If $g(x)$ and $h(x)$ have the same power, the horizontal asymptote is
$$y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$$
3) If $g(x)$ is a higher power than $h(x)$, there is no horizontal asymptote.

Let's investigate all three cases:

a) $f(x) = \frac{1}{x-2}$ b) $f(x) = \frac{1}{x-2} + 3$ c) $f(x) = \frac{2x}{x^2-4}$ d) $\frac{3x+1}{x-2}$ e) $y = \frac{3x(2x-1)}{4x^2-1}$ f) $y = \frac{x^2}{x}$

a) $h(x)$ higher power than $g(x)$, $K = 0$, so horiz asymp is $y = 0$

b) $h(x)$ higher power than $g(x)$, $k = 3$, horiz asymp $y = 3$

c) $h(x)$ higher, $k = 0$, horiz asymp $y = 0$

d) Same powers so horiz asymp $y = \frac{3}{1} = 3$

e) $y = \frac{6x^2 - 3x}{4x^2 - 1}$, horiz asymp $y = \frac{6}{4} = \frac{3}{2}$

f) $h(x)$ lower degree than $g(x)$, so no horiz asymp

$y = \frac{a}{x-h} + k$

4.5B – Rational Functions Part 2

Steps:

1) H.A.

2) V.A.

3) Test x values one unit left/right of V.A.

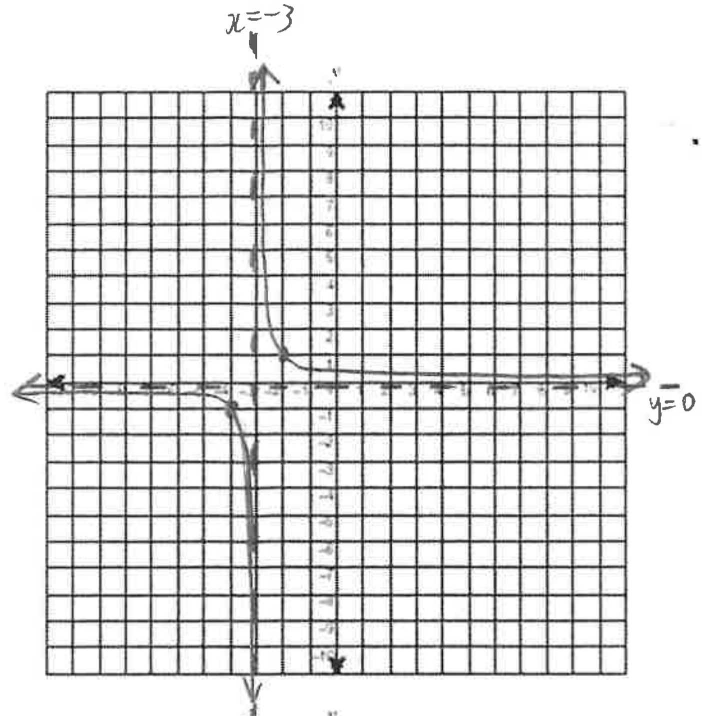
4) asymptotes get approached

Example 1 – Graph $f(x) = \frac{1}{x+3}$

x	y
-4	-1
-2	1
$(-3)^-$	$-\infty$
$(-3)^+$	∞
-1000	0^-
1000	0^+

H.A. $y=0$
V.A. $x=-3$

$a=1$
right 1 up 1
left 1 down 1
- approach asymptotes



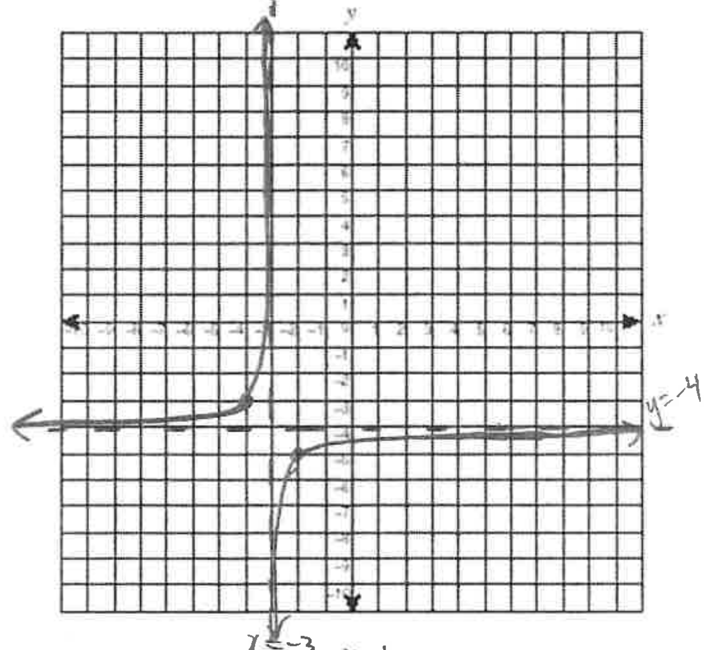
D: $x \neq -3$ R: $y \neq 0$

x-int: $\emptyset$ y-int: $y = \frac{1}{3}$

Graph $f(x) = -\frac{1}{x+3} - 4$

H.A. $y=-4$
V.A. $x=-3$

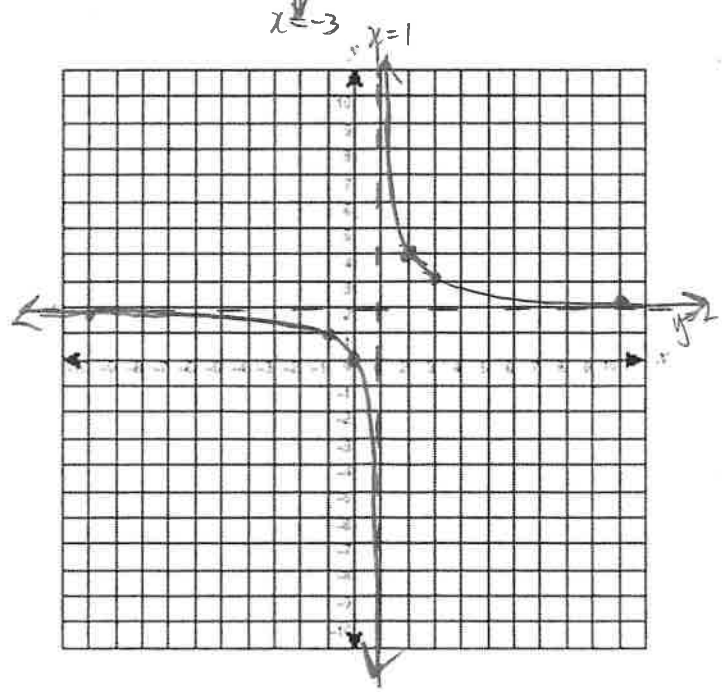
(note $a=-1$)
right 1 down 1
left 1 up 1
- approach asymptotes



Example 2 – Graph $y = \frac{2x}{x-1}$

H.A. $y=2$
V.A. $x=1$

x	y
0	0
2	4
10	$2.\bar{2}$
-10	1.8
1.01	202



D: $x \neq 1$ R: $y \neq 2$

x-int: 0 y-int: 0

Steps:

1) H.A.

2) V.A.

3) If no V.A, test a bunch of x values

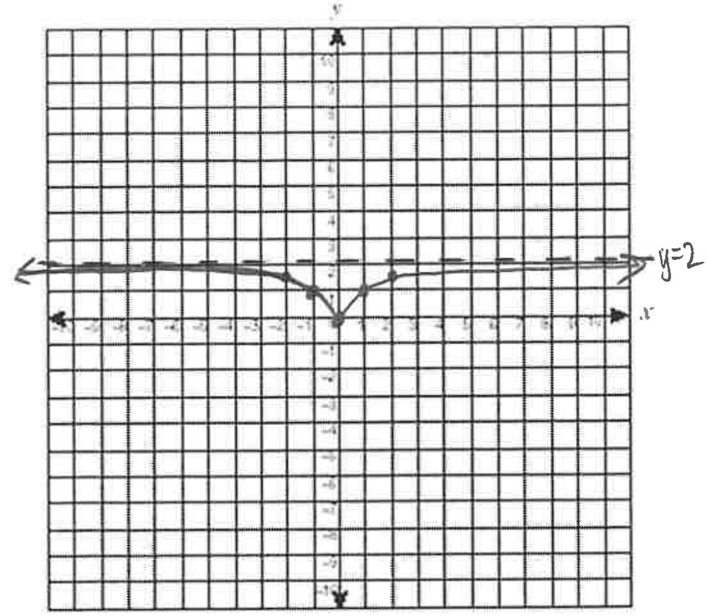
4) asymptotes get approached

Example 3 – Graph $h(x) = \frac{2x^2}{x^2+1}$

H.A. $y = \frac{2}{1} = 2$

no V.A.

x	y
0	0
1	1
−1	1
2	1.6
−2	1.6
10	1.98
−10	1.98



D: $x \in \mathbb{R}$ *R:* $0 \leq y < 2$ *x-int:* 0 *y-int:* 0

Steps:

1) H.A.

2) V.A.

3) If two V.A & one H.A., there are four common possibilities (see next page)

4) Test far left and far right (one unit from V.A.)

5) Test middle: (one unit from each V.A. and exact middle x value

6) asymptotes get approached

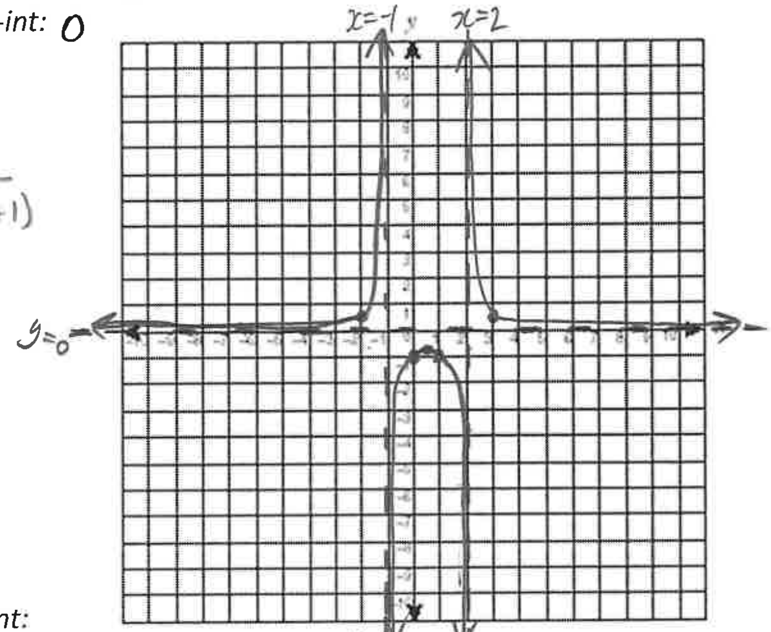
Example 4 – Graph $f(x) = \frac{2}{x^2-x-2}$

$= \frac{2}{(x-2)(x+1)}$

H.A. $y=0$

V.A. $x = -1, 2$

x	y
3	$\frac{1}{2}$
−2	$\frac{1}{2}$
0	−1
1	−1
0.5	$-\frac{8}{9}$
−0.9	−6.9
1.9	−6.9



D: $x \neq -1, 2$ *R:* $y > 0$, $y \leq -\frac{8}{9}$ *x-int:* $\emptyset$ *y-int:* −1

Example 5 – Graph $y = \frac{x^2-3x-4}{x^2+2x} = \frac{(x-4)(x+1)}{x(x+2)}$

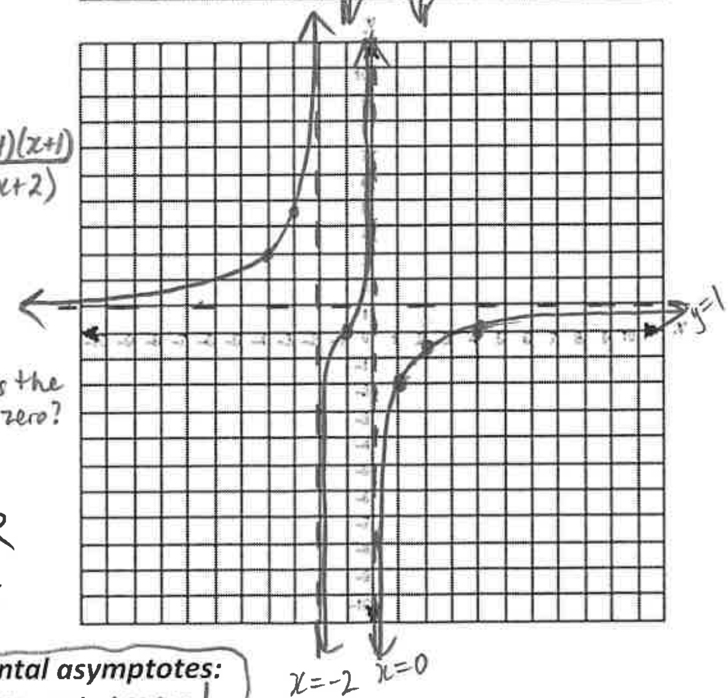
H.A. $y = \frac{1}{1} = 1$

V.A. $x = -2, 0$

$\boxed{x\text{-int}}$

$0 = \frac{(x-4)(x+1)}{x(x+2)}$ What makes the numerator zero? −1, 4

x	y
1	−2
2	−0.75
−3	4.7
−4	3
−1	0
−1.9	−28
−0.1	19.4

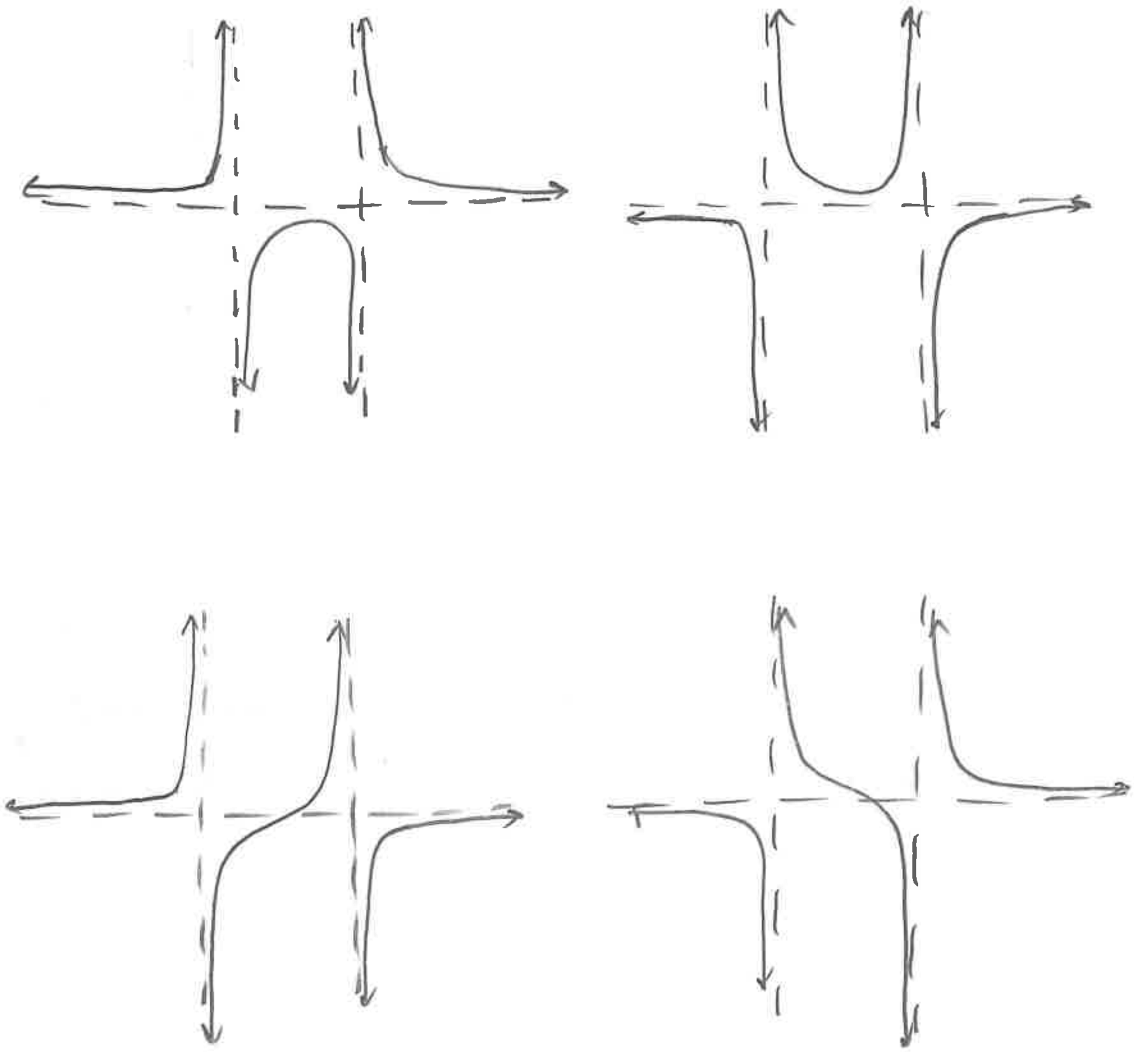


D: $x \neq -2, 0$ *R:* $y \in \mathbb{R}$

x-int: −1, 4 *y-int:* $\emptyset$

A note about horizontal asymptotes:
In some graphs, they are crossed.

Four common possibilities with two V.A. and one H.A.:



Rational Functions & Graphing Part 3 – 'Holes'

Sometimes, due to factoring and cancellation, a rational function simplifies, which can cause cancellation of parts of the denominator, resulting in the elimination of a vertical asymptote. This will create a 'hole' in the graph of the function.

Graph $y = \frac{x+3}{x^2+x-6}$ $\quad x \neq -3, 2$

$y = \frac{x+3}{(x+3)(x-2)}$

$y = \frac{1}{x-2}$

Asymps cross at $(2,0)$

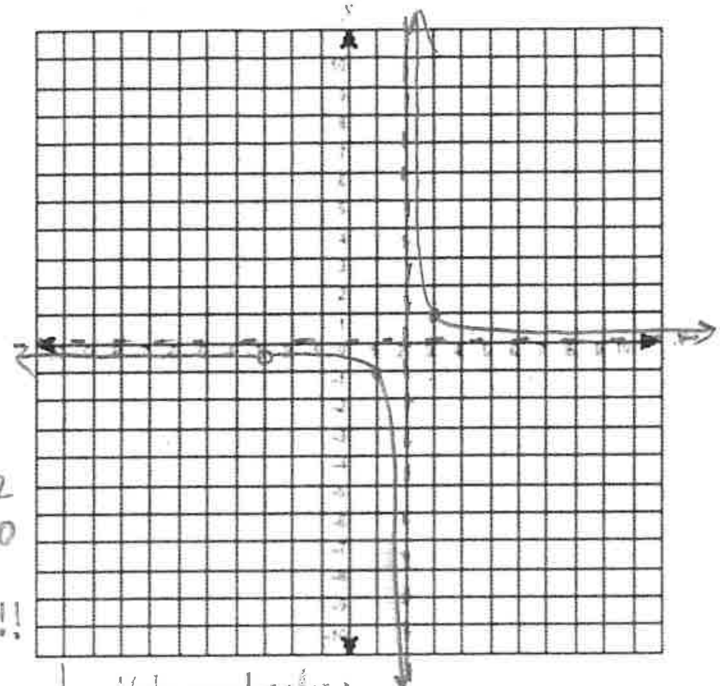
Same shape as $y = \frac{1}{x}$

x	y
1	−1
3	1
−3	undef!

vert asymp: $x = 2$

horiz asymp: $y = 0$

← hole at $x = -3$!!



$D: x \neq -3, 2$

$R: y \neq 0, -\frac{1}{5}$

Hole coordinates:

$x = -3$

$y = \frac{1}{-3-2} = -\frac{1}{5}$

$\left(-3, -\frac{1}{5}\right)$

Where there would have been a vertical asymptote, there is now just a 'hole' in the graph.

For each of the following functions, determine any vertical asymptotes and the coordinates of any holes

a) $y = \frac{x+4}{x^2+3x-4}$

$y = \frac{x+4}{(x+4)(x-1)}$ $\quad x \neq -4, 1$

$y = \frac{1}{x-1}$

hole at $(-4, -)$

$y = \frac{1}{-4-1} = -\frac{1}{5}$

hole at $\left(-4, -\frac{1}{5}\right)$

b) $y = \frac{x^2+2x-3}{x-1}$ $\quad x \neq 1$

$y = \frac{(x+3)(x-1)}{x-1}$

$y = x + 3$ (just a line w/ $m=1$, $b=3$)

hole at $(1, -)$

$y = 1 + 3 = 4$

hole at $(1, 4)$

c) $f(x) = \frac{x}{x^2-2x}$

$y = \frac{x}{x(x-2)}$ $\quad x \neq 0, 2$

$y = \frac{1}{x-2}$

hole at $(0, -)$

$y = \frac{1}{0-2} = -\frac{1}{2}$

hole at $\left(0, -\frac{1}{2}\right)$

Graph the function & state any *asymptotes, hole coordinates, domain, range, x-int, y-int:*

a) $y = \frac{x^2-2x-8}{x+2}$ H.A none

$y = \frac{(x-4)\cancel{(x+2)}}{\cancel{x+2}}$ V.A. none

$y = \frac{1}{1}x - 4$ Hole: $(-2,\ )$

↑

straight line

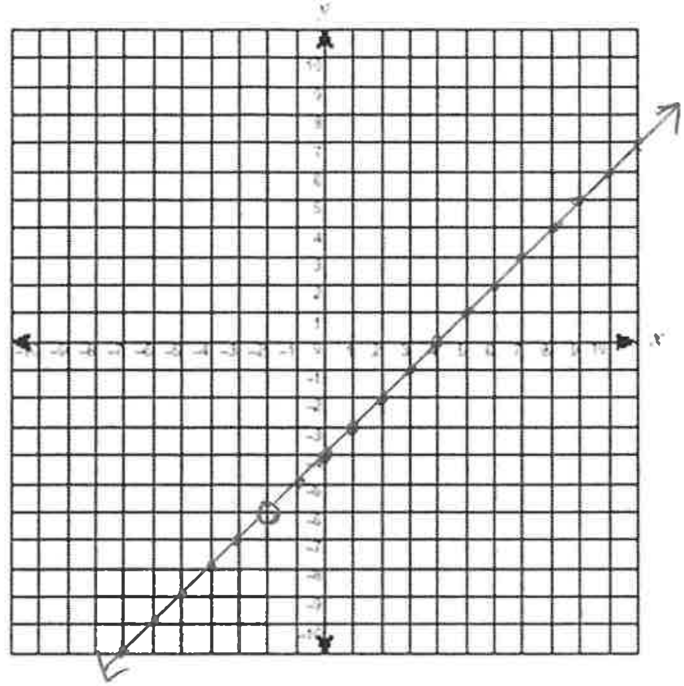
y-int = -4

slope = $\frac{1 \leftarrow \text{up}}{1 \leftarrow \text{right}}$

$y = -2-4$

$y = -6$

Hole $(-2,-6)$



D: $x \neq -2$

R: $y \neq -6$

x-int 4

y-int: -4

Hole coordinates $(-2,-6)$

b) $f(x) = -\frac{x-4}{x^2-x-12}$

$y = -\frac{\cancel{x-4}}{\cancel{(x-4)}(x+3)}$

$y = \frac{-1}{x+3}$ H.A $y=0$ V.A $x=-3$

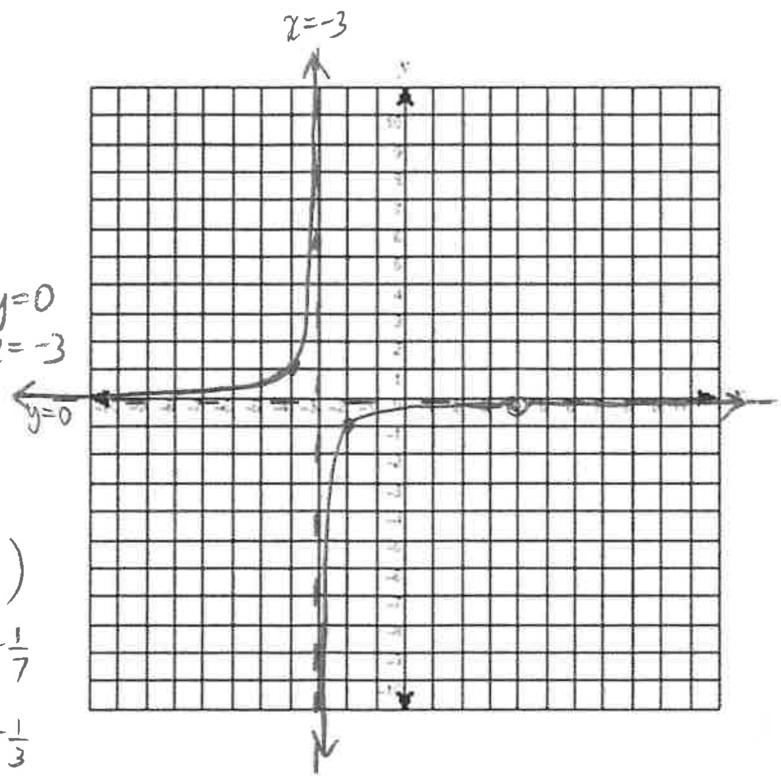
$a=-1$

right 1 down 1

left 1 up 1

Hole coordinates $(4, \frac{-1}{7})$

$y = \frac{-1}{4+3} = -\frac{1}{7}$



D: $x \neq -3, 4$

R: $y \neq -\frac{1}{7}, 0$

x-int: $\emptyset$

y-int: $-\frac{1}{3}$

$y = \frac{-1}{0+3} = \frac{-1}{3}$