

## 5.1 – Solving Quadratic Equations by Factoring

A function of *degree 2* (meaning the highest exponent on the variable is 2) is called a **Quadratic Function**.

**Quadratic functions** are written as, for example,  $f(x) = x^2 - x - 6$  OR  $y = x^2 - x - 6$ .  $f(x)$  is 'f of x' and means that the y value is dependent upon the value of x. Once you have an x value and you substitute it into the function, the value of  $f(x)$  will result.  $f(x)$  is really just another way of writing y, and is used when the graph is a function (passes the vertical line test).

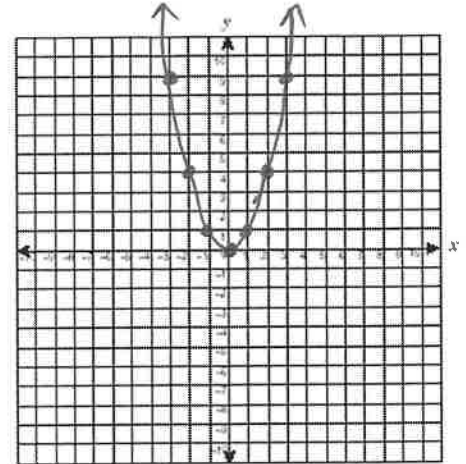
When a quadratic function is graphed, a **parabola** results.

Example 1:  $y = x^2$

$$y = (-3)^2$$

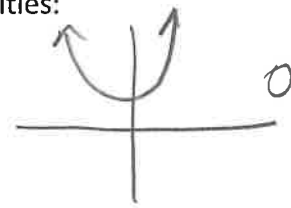
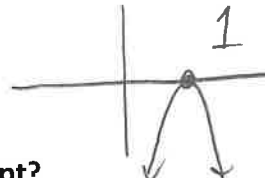
| x  | y |
|----|---|
| -3 | 9 |
| -2 | 4 |
| -1 | 1 |
| 0  | 0 |
| 1  | 1 |
| 2  | 4 |
| 3  | 9 |

$(-3, 9)$   
 $(-2, 4)$   
 etc.



A very important feature of the parabola that results from a quadratic function is where it touches or crosses the x-axis. These are the **x-intercepts** of the parabola.

How many **x-intercepts** can a parabola have? Draw all possibilities:



**What is the y value at an x-intercept?**

0 ;  $(-, 0)$  on x axis

Therefore, to find the x-intercepts of a parabola, we can set  $y = 0$  (or  $f(x) = 0$ ) and solve the resulting **quadratic equation**.

When y is set to 0, we call the question a **quadratic equation** instead of a quadratic function.

Quadratic function:  $f(x) = x^2 - x - 6$  OR  $y = x^2 - x - 6$

Quadratic equation:  $x^2 - x - 6 = 0$

The **x-intercepts** of the parabola are the **zeros** of the quadratic *function*. They are also called the **solutions** or **roots** of the quadratic *equation*.

Therefore, how many **zeros** can there be for a quadratic function? 0, 1, 2

How many **roots** or **solutions** can there be for a quadratic equation? 0, 1, 2

One method to find the **zeros** of a **quadratic function** is to graph it and visually determine the **x-intercepts**.

You can often find the **roots** of a **quadratic equation** by factoring when in general form  $ax^2 + bx + c = 0$ . Remember, the **roots** or **solutions** of the quadratic equation correspond to the **zeros** of the quadratic function, and the **x-intercepts** of the parabola.

if  $a = 1$

Example 2 - Solve and check  $x^2 + 3x = -2$

To 'solve' a quadratic equation means to find the **roots** or **solutions**. Steps are as follows:

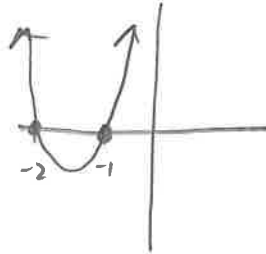
- 1) Get everything to one side so that only zero is on the other.
- 2) Identify  $a$ ,  $b$ , and  $c$  values, and factor accordingly into binomials.
- 3) The roots are the  $x$ -values that will make the product of the binomials zero. If either of the binomials equal zero, then the product of the binomials will equal zero (this is called the **Zero Factor Property**). Therefore, identify the  $x$  values that make each binomial equal to zero.

Solve and check  $x^2 + 3x = -2$

$$x^2 + 3x + 2 = 0 \quad \begin{array}{l} \times 2 \\ + 3 \\ 2, 1 \end{array}$$

$$(x+2)(x+1) = 0$$

$$x = -2, -1$$



Sketch the Graph:

Check:  $x = -2$

|                  |      |
|------------------|------|
| LS               | RS   |
| $x^2 + 3x$       | $-2$ |
| $(-2)^2 + 3(-2)$ |      |
| $4 + -6$         |      |
| $-2$             |      |

$x = -1$

|                  |      |
|------------------|------|
| LS               | RS   |
| $x^2 + 3x$       | $-2$ |
| $(-1)^2 + 3(-1)$ |      |
| $1 - 3$          |      |
| $-2$             |      |

Example 3: a) Solve  $x^2 - 8x - 40 = 8$

$$x^2 - 8x - 48 = 0$$

$$(x-12)(x+4) = 0$$

$$x = -4, 12$$

b) Solve  $2x^2 + 6x - 108 = 0$

$$\frac{2}{2}(x^2 + 3x - 54) = \frac{0}{2}$$

$$x^2 + 3x - 54 = 0$$

$$(x+9)(x-6) = 0$$

$$x = -9, 6$$

c) Solve  $\frac{1}{2}x^2 - x - 4 = 0$

$$\frac{1}{2}x^2 - x - 4 = 0 \quad (2)$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = -2, 4$$

d) Solve  $\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{x^2 - 4x - 5}$

$$\frac{x(x-5)(x+1)}{x-5} - \frac{3(x-5)(x+1)}{x+1} = \frac{30(x-5)(x+1)}{(x-5)(x+1)}$$

$$x(x+1) - 3(x-5) = 30$$

$$x^2 + x - 3x + 15 = 30$$

$$x^2 - 2x - 15 = 0$$

$$(x-5)(x+3) = 0$$

$$x = -3, 5$$

*extraneous*

$$x = -3$$

if  $a \neq 1$

When  $a \neq 1$ , factor by "decomposition."

Example 4 - Solve  $3x^2 - 5x + 2 = 0$

$$3x^2 - 3x - 2x + 2 = 0$$

$$3x(x-1) - 2(x-1) = 0$$

$$(x-1)(3x-2) = 0$$

$$x = \frac{2}{3}, 1$$

$\times 6$   
 $+ -5$   
 $-3, -2$

Trick for finding roots:

constant over  
coefficient,  
switch the sign

Example 5 - a) Solve  $3(x^2 - 2) = -7x$

$$3x^2 - 6 = -7x$$

$$3x^2 + 7x - 6 = 0$$

$$3x^2 + 9x - 2x - 6 = 0$$

$$3x(x+3) - 2(x+3) = 0$$

$$(x+3)(3x-2) = 0$$

$$x = -3, \frac{2}{3}$$

b) Solve  $2(3x+2)^2 - 5(3x+2) = -2$

$$2(3x+2)^2 - 5(3x+2) + 2 = 0$$

$$\text{Let } m = 3x+2$$

$$2m^2 - 5m + 2 = 0$$

$$2m^2 - 4m - m + 2 = 0$$

$$2m(m-2) - 1(m-2) = 0$$

$$(m-2)(2m-1) = 0$$

$$m = \frac{1}{2}, 2$$

$$3x+2=2, 3x+2=\frac{1}{2}$$

$$3x=0 \quad 3x=-\frac{3}{2}$$

$$x=0 \quad x=-\frac{1}{2}$$

$$x = 0, -\frac{1}{2}$$

if  $c = 0$

Example 6 - Solve  $3x^2 = -5x$

$$3x^2 + 5x$$

$$x(3x+5) = 0$$

$$x = 0, -\frac{5}{3}$$

difference  
of squares

Example 7 - a) Solve  $x^2 - 25 = 0$

$$(x+5)(x-5) = 0$$

$$x = \pm 5$$

b) Solve  $8p^2 - 18 = 0$

$$\frac{2(4p^2 - 9)}{2} = \frac{0}{2}$$

$$4p^2 - 9 = 0$$

$$(2p+3)(2p-3) = 0$$

$$p = \pm \frac{3}{2}$$

c) Solve  $49 - 4x^2 = 0$

$$(7+2x)(7-2x) = 0$$

$$x = \pm \frac{7}{2}$$

Example 8 - a) Write a quadratic equation with roots 6 and -1. b)  $\frac{2}{3}$  and  $-\frac{1}{2}$

$$\text{a) } (x-6)(x+1) = 0$$

$$x^2 - 5x - 6 = 0$$

$$\text{b) } (3x-2)(2x+1) = 0$$

$$6x^2 - x - 2 = 0$$

When a radical is simplified, an *entire radical* is changed to a *mixed radical* (or a mixed radical is further simplified). Example 1 – Simplify

$$\begin{aligned} \text{a) } \sqrt{18} \\ \sqrt{9 \cdot 2} \\ \sqrt{9} \cdot \sqrt{2} \\ 3\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{b) } \sqrt{24} \\ \sqrt{4 \cdot 6} \\ \sqrt{4} \cdot \sqrt{6} \\ 2\sqrt{6} \end{aligned}$$

$$\begin{aligned} \text{c) } 3\sqrt{50} \\ 3\sqrt{25 \cdot 2} \\ 3 \cdot \sqrt{25} \cdot \sqrt{2} \\ 3 \cdot 5 \cdot \sqrt{2} \\ 15\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{d) } \sqrt{72} \\ \sqrt{36 \cdot 2} \\ \sqrt{36} \cdot \sqrt{2} \\ 6\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{e) } 2\sqrt{45} \\ 2\sqrt{9 \cdot 5} \\ 2 \cdot \sqrt{9} \cdot \sqrt{5} \\ 2 \cdot 3 \cdot \sqrt{5} \\ 6\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{f) } \sqrt{50} \\ \sqrt{25 \cdot 2} \\ 5\sqrt{2} \end{aligned}$$

A final answer cannot have a radical in the denominator. Therefore, you may have to '**rationalize the denominator**' – a process that will eliminate the radical from the denominator without changing the value of the expression.

Example 2 - Simplify:

$$\begin{aligned} \text{a) } \frac{3 \cdot \sqrt{11}}{\sqrt{11} \cdot \sqrt{11}} & \quad \text{b) } \frac{\sqrt{2}}{\sqrt{5}} & \quad \text{c) } \frac{7 \cdot \sqrt{3}}{2\sqrt{3} \cdot \sqrt{3}} \\ \frac{3\sqrt{11}}{11} & \quad \frac{\sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} & \quad \frac{7\sqrt{3}}{6} \\ & \quad \frac{\sqrt{10}}{5} & \end{aligned}$$

Sometimes, you can further simplify after rationalizing the denominator.

Example 3 – Simplify:

$$\begin{aligned} \text{a) } \frac{4 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} & \quad \text{b) } \frac{6\sqrt{3} \div 3}{\sqrt{18} \div 3} & \quad \text{c) } \frac{7\sqrt{6} \div 2}{2\sqrt{10} \div 2} \\ \frac{4\sqrt{2}}{2} & \quad \frac{6\sqrt{1}}{\sqrt{6}} & \quad \frac{7\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}} \\ 2\sqrt{2} & \quad \frac{6 \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} & \quad \frac{7\sqrt{15}}{10} \\ & \quad \frac{6\sqrt{6}}{6} = \sqrt{6} & \end{aligned}$$

Sometimes, you can first simplify by dividing, and then you may or may not have to rationalize the denominator after. When dividing radicals, divide the coefficients, and divide the radicands.

Example 4 – Simplify a)  $\frac{4\sqrt{14}}{2\sqrt{2}}$

$$2\sqrt{7}$$

b)  $\frac{-9\sqrt{48}}{6\sqrt{3}}$

$$\frac{-3\sqrt{16}}{2}$$

$$\frac{-3(4)}{2}$$

$$= -6$$

c)  $\frac{4\sqrt{5 \div 5}}{\sqrt{35 \div 5}}$

$$\frac{4 \cdot \sqrt{1}}{\sqrt{7} \cdot \sqrt{1}}$$

$$\frac{4\sqrt{1}}{7}$$

When solving quadratic equations using the quadratic formula (section 3.3), you will often have exact answers that may need simplifying. Here are some examples:

Example 5 – Simplify a)  $\frac{-3 \pm \sqrt{27}}{6}$

$$\frac{-3 \pm 3\sqrt{3}}{6 \div 3}$$

$$\frac{-1 \pm \sqrt{3}}{2}$$

b)  $\frac{2 \pm \sqrt{48}}{2}$

$$\frac{2 \pm \sqrt{16 \cdot 3}}{2}$$

$$\frac{2 \pm 4\sqrt{3}}{2 \div 2}$$

$$1 \pm 2\sqrt{3}$$

c)  $\frac{5 \pm \sqrt{8}}{10}$

$$\frac{5 \pm \sqrt{4 \cdot 2}}{10}$$

$$\frac{5 \pm 2\sqrt{2}}{10}$$

## 5.2A – Solving Quadratic Equations by Completing the Square

square  
root  
principle

When there is no  $bx$  term in a quadratic equation, it may be *difference of squares* (is each term a perfect square, and is there a subtraction sign in between?), and could be solved by factoring. But any quadratic equation without a  $bx$  term can be solved using the *square root principle*. The general form of a quadratic equation that you could solve using the square root principle is  $ax^2 \pm c = 0$ .

Example 1 – Solve  $9y^2 - 21 = 0$

- 1) Get everything that isn't squared to one side.
- 2) Square root both sides.
- 3) Consider both the positive and negative square roots.
- 4) Simplify any radical solutions as much as possible in exact form.

$$\begin{array}{l|l} 9y^2 - 21 = 0 & y = \pm \frac{\sqrt{21}}{\sqrt{9}} \\ 9y^2 = 21 & \\ y^2 = \frac{21}{9} & y = \pm \frac{\sqrt{21}}{3} \\ \sqrt{y^2} = \pm \sqrt{\frac{21}{9}} & \end{array}$$

Example 2 - Solve: a)  $2x^2 - 11 = 87$

b)  $50y^2 = 72$

c)  $(x+3)^2 = 16$

a)  $2x^2 - 11 = 87$

$$2x^2 = 98$$

$$x^2 = 49$$

$$x = \pm \sqrt{49}$$

$$x = \pm 7$$

b)  $50y^2 = 72$

$$y^2 = \frac{72}{50}$$

$$y^2 = \frac{36}{25}$$

$$y = \pm \frac{\sqrt{36}}{\sqrt{25}}$$

$$y = \pm \frac{6}{5}$$

c)  $x+3 = \pm \sqrt{16}$

$$x+3 = \pm 4$$

$$x+3 = 4 \quad | \quad x+3 = -4$$

$$x = 1 \quad | \quad x = -7$$

$$x = -7, 1$$

d)  $3x^2 - 8 = 0$

$$3x^2 = 8$$

$$x^2 = \frac{8}{3}$$

$$x = \pm \frac{\sqrt{8}}{\sqrt{3}}$$

$$x = \pm \frac{2\sqrt{2} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$

$$x = \pm \frac{2\sqrt{6}}{3}$$

e)  $(x-1)^2 = 12$

$$x-1 = \pm \sqrt{12}$$

$$x-1 = \pm 2\sqrt{3}$$

$$x = 1 \pm 2\sqrt{3}$$

completing the square when  $a = 1$

Sometimes factoring quadratic equations (5.1) is not possible, as you cannot find the two numbers that multiply to  $c$  (or  $ac$ ) and add to  $b$ . When this is the case, you can still solve the quadratic equation by a method called *completing the square*.

Example 3 – Solve  $x^2 - 24 = -10x$  by completing the square. This example can be easily solved by factoring, but we will use it to introduce how to complete the square.

- 1) Get  $c$  to one side of the equation.
- 2) If the  $a$  value is 1, find the  $b$  value, half it, and square it.
- 3) Add the number from step 2 to BOTH sides of the equation.
- 4) Factor the trinomial on the left (we created a perfect square trinomial, so it will lead to 2 brackets that are exactly the same).
- 5) Solve using the square root property.

$$\begin{array}{l}
 x^2 - 24 = -10x \\
 x^2 + 10x = 24 \\
 b = 10, 5, 25 \\
 x^2 + 10x + 25 = 24 + 25 \\
 (x+5)^2 = 49
 \end{array}
 \left|
 \begin{array}{l}
 x+5 = \pm\sqrt{49} \\
 x+5 = 7 \quad | \quad x+5 = -7 \\
 x = 2 \quad | \quad x = -12 \\
 x = -12, 2
 \end{array}
 \right.$$

Example 4 – Solve by completing the square. Express the solutions in exact form.

|                         |   |   |
|-------------------------|---|---|
| a) $w^2 - 4w - 11 = 0$  | b) $x^2 + 5x + 7 = 0$                                     | c) $m^2 - 5m + 3 = 0$                                     |
| $w^2 - 4w = 11$         | $x^2 + 5x = -7$   | $m^2 - 5m = -3$   |
| $b = -4, -2, 4$         | $b = 5, \frac{5}{2}, \frac{25}{4}$                        | $b = -5, -\frac{5}{2}, \frac{25}{4}$                      |
| $w^2 - 4w + 4 = 11 + 4$ | $x^2 + 5x + \frac{25}{4} = \frac{-7}{1.4} + \frac{25}{4}$ | $m^2 - 5m + \frac{25}{4} = \frac{-3}{1.4} + \frac{25}{4}$ |
| $(w-2)^2 = 15$          | $(x + \frac{5}{2})^2 = \frac{-3}{4}$                      | $(m - \frac{5}{2})^2 = \frac{13}{4}$                      |
| $w - 2 = \pm\sqrt{15}$  | $x + \frac{5}{2} = \pm\sqrt{\frac{-3}{4}}$                | $m - \frac{5}{2} = \pm\sqrt{\frac{13}{4}}$                |
| $w = 2 \pm\sqrt{15}$    | $x + \frac{5}{2} = \pm\frac{\sqrt{-3}}{2}$                | $m - \frac{5}{2} = \pm\frac{\sqrt{13}}{2}$                |
|                         | $\emptyset$ undefined                                     | $m = \frac{5 \pm \sqrt{13}}{2}$                           |

## 5.2B – Solving Quadratic Equations by Completing the Square Part 2

completing  
the square  
when  $a \neq 1$

If  $a \neq 1$ , there are a few more considerations when *completing the square*.

Example 1 – Solve  $2x^2 - 5x - 1 = 0$  by completing the square

- 1) Get  $c$  to one side of the equation.
- 2) Factor the  $a$  value out of the left side.
- 3) Divide both sides by the  $a$  value to leave  $x^2 + bx$  on the left side of the equation.  
THEN find the  $b$  value, half it, and square it.
- 4) Add the number from step 3 to BOTH sides of the equation.
- 5) Factor the resulting trinomial on the left.
- 6) Solve using the square root principle and answer in exact form.

$$\begin{array}{l}
 2x^2 - 5x - 1 = 0 \\
 2x^2 - 5x = 1 \\
 2(x^2 - \frac{5}{2}x) = 1 \\
 x^2 - \frac{5}{2}x = \frac{1}{2} \\
 b = \frac{-5}{2}, \frac{-5}{4}, \frac{25}{16}
 \end{array}
 \left|
 \begin{array}{l}
 x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{1}{2} + \frac{25}{16} \\
 (x - \frac{5}{4})^2 = \frac{33}{16} \\
 x - \frac{5}{4} = \pm \frac{\sqrt{33}}{\sqrt{16}} \\
 x - \frac{5}{4} = \pm \frac{\sqrt{33}}{4}
 \end{array}
 \right|
 \quad x = \frac{5 \pm \sqrt{33}}{4}$$

Example 2 – Solve by completing the square. Answer  $b$  to the nearest hundredth.

a)  $3x^2 - 2 = -4x$

$$\begin{array}{l}
 3x^2 + 4x = 2 \\
 3(x^2 + \frac{4}{3}x) = 2 \\
 x^2 + \frac{4}{3}x = \frac{2}{3} \\
 b = \frac{4}{3}, \frac{4}{6} = \frac{2}{3}, \frac{4}{9} \\
 x^2 + \frac{4}{3}x + \frac{4}{9} = \frac{2}{3} + \frac{4}{9} \\
 (x + \frac{2}{3})^2 = \frac{10}{9} \\
 x + \frac{2}{3} = \pm \frac{\sqrt{10}}{\sqrt{9}}
 \end{array}$$

b)  $-2x^2 - 3x + 7 = 0$

$$\begin{array}{l}
 2x^2 + 3x = 7 \\
 2(x^2 + \frac{3}{2}x) = 7 \\
 x^2 + \frac{3}{2}x = \frac{7}{2} \\
 b = \frac{3}{2}, \frac{3}{4}, \frac{9}{16} \\
 x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{7}{2} + \frac{9}{16} \\
 (x + \frac{3}{4})^2 = \frac{65}{16} \\
 x + \frac{3}{4} = \pm \frac{\sqrt{65}}{4} \\
 x = \frac{-3 \pm \sqrt{65}}{4} \quad // \quad x = 1.27, -2.77
 \end{array}$$

An easy way to half a fraction is to double the denominator.



Example 3 – Solve by completing the square:  $3x^2 + 6x - 1 = 0$

$$3x^2 + 6x = 1$$

$$3(x^2 + 2x) = 1$$

$$x^2 + 2x = \frac{1}{3}$$

$$b = 2, 1, 1$$

$$x^2 + 2x + 1 = \frac{1}{3} + \frac{1 \cdot 1}{1 \cdot 3}$$

$$(x+1)^2 = \frac{4}{3}$$

$$x+1 = \pm \frac{\sqrt{4}}{\sqrt{3}}$$

$$x+1 = \pm \frac{2}{\sqrt{3}}$$

$$x = -1 \pm \frac{2 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$

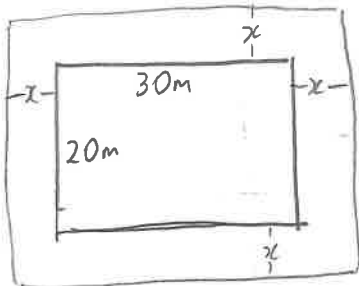
$$x = \frac{-1 \pm 2\sqrt{3}}{1 \cdot 3}$$

$$x = \frac{-3 \pm 2\sqrt{3}}{3}$$

word  
problems

Example 4 - Butchart Gardens wants to build a pathway around its rose garden. The rose garden is currently 30m x 20m. The pathway will be built by extending each side by an equal amount. If the area of the garden and path together is 1.173 times larger than the area of just the rose garden, how wide will the new path be (to the nearest tenth)?

Let  $x =$  width of path



$$A_{\text{rose garden}} = (30)(20) = 600 \text{m}^2$$

$$A_{\text{total}} = 600(1.173) = 703.8 \text{m}^2$$

$$(30+2x)(20+2x) = 703.8$$

$$4x^2 + 100x + 600 = 703.8$$

$$4x^2 + 100x = 103.8$$

$$4(x^2 + 25x) = 103.8$$

$$x^2 + 25x = 25.95$$

$$b = 25, 12.5, 156.25$$

$$x^2 + 25x + 156.25 = 25.95 + 156.25$$

$$(x+12.5)^2 = 182.2$$

$$x+12.5 = \pm 13.5$$

$$x+12.5 = 13.5 \quad | \quad x+12.5 = -13.5$$

$$x = 1.0 \quad | \quad x = \frac{-26}{2} \text{ reject}$$

The pathway will be 1.0m wide.

### 5.3 – The Quadratic Formula

Quadratic equations can be solved by graphing, factoring (5.1), or completing the square (5.2). Each of these methods have advantages and limitations.

Any quadratic equation can be solved using something called the *quadratic formula*. If the quadratic equation is in general form ( $ax^2 \pm bx \pm c = 0$ ), the quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1 - Solve  $3x^2 + 5x - 2 = 0$  using the quadratic formula

$$a=3 \quad b=5 \quad c=-2$$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

$$x = \frac{-5 \pm \sqrt{49}}{6}$$

$$x = \frac{-5 \pm 7}{6}$$

$$x = \frac{2}{6} \quad \text{or} \quad \frac{-12}{6}$$

$$x = \frac{1}{3}, -2$$

discriminant

There could be 0, 1, or 2 resulting roots, depending on the **discriminant**, the expression under the square root ( $b^2 - 4ac$ ). Solutions can be written in simplest radical form, or decimal form.

How would the discriminant determine the *nature of the roots* (the number of roots)?

If  $b^2 - 4ac > 0$ , there will be two solutions

If  $b^2 - 4ac = 0$ , there will be one solution

If  $b^2 - 4ac < 0$ , there will be no solutions.

quadratic formula

Example 2 – Determine the nature of the roots, and then solve  $3x^2 + 2x - 4 = 0$  using the quadratic formula

$$a=3 \quad b=2 \quad c=-4$$

how many solutions? 0, 1, or 2?

$$b^2 - 4ac$$

$$2^2 - 4(3)(-4)$$

$$4 + 48$$

$$52$$

Since the discriminant is positive, there are two solutions.

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(-4)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{52}}{6}$$

$$x = \frac{-2 \pm 2\sqrt{13}}{6}$$

$$x = \frac{-1 \pm \sqrt{13}}{3}$$

Example 3 – Determine the nature of the roots for  $\frac{1}{4}x^2 - 3x + 9 = 0$

$$(4) \frac{1}{4}x^2 - 3x + 9 = 0$$

$$x^2 - 12x + 36 = 0$$

$$a=1 \quad b=-12 \quad c=36$$

$$b^2 - 4ac$$

$$(-12)^2 - 4(1)(36)$$

$$144 - 144$$

$$0$$

Because the discriminant is zero, there is only one solution.

Example 4 – Solve using the quadratic formula. Leave answers in exact form.

a)  $x^2 = 2x + 1$

$$x^2 - 2x - 1 = 0$$

$$a=1, b=-2, c=-1$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = \frac{2 \pm 2\sqrt{2}}{2}$$

$$x = 1 \pm \sqrt{2}$$

derivation

Derive the quadratic formula:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$a(x^2 + \frac{b}{a}x) = -c$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

$$\frac{b}{a}, \frac{b}{2a}, \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

b)  $\frac{6x^2}{2} - \frac{5x}{6} = \frac{-3}{2}$

$$3x^2 - 5x = -9$$

$$3x^2 - 5x + 9 = 0$$

$$a=3, b=-5, c=9$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(9)}}{2(3)}$$

$$x = \frac{5 \pm \sqrt{25 - 108}}{6}$$

$$x = \frac{5 \pm \sqrt{-83}}{6} \text{ therefore } \emptyset$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## 2.5 – Rational Equations (Quadratic)

A rational equation is an equation containing at least one rational expression. Remember, when working with an equation, whatever you do to one side, you do to the other side.

Steps to solving rational equations:

- 1) Factor each denominator if possible.
- 2) Identify any undefined values (and do this throughout).
- 3) Multiply both sides of the equation by what would be the lowest common denominator in order to eliminate the fractions.
- 4) Solve the equation.
- 5) Check your solutions.

Example 1 – Solve a)  $\frac{x}{4} + \frac{6}{x} = 3.5$   $x \neq 0$

$$\frac{(4x)x}{4} + \frac{6(4x)}{x} = 3.5(4x)$$

$$x^2 + 24 = 14x$$

$$x^2 - 14x + 24 = 0$$

$$(x-12)(x-2) = 0$$

$$x = 2, 12$$

b)  $3 - \frac{6}{x} = x + 8$   $x \neq 0$

$$3x - 6 = x^2 + 8x$$

$$x^2 + 5x + 6 = 0$$

$$(x+3)(x+2) = 0$$

$$x = -3, -2$$

Example 2 – Solve  $\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{x^2-4x-5}$   $x \neq -1, 5$

$$\frac{x}{x-5} - \frac{3}{x+1} = \frac{30}{(x-5)(x+1)}$$

$$\frac{x(x+1)}{x-5} - \frac{3(x-5)}{x+1} = \frac{30(x-5)(x+1)}{(x-5)(x+1)}$$

$$x(x+1) - 3(x-5) = 30$$

$$x^2 + x - 3x + 15 = 30$$

$$x^2 - 2x - 15 = 0$$

$$(x-5)(x+3) = 0$$

$$x = -3, \text{ extraneous}$$

$$x = -3$$

Example 3 - Solve  $\frac{3x}{x+2} - \frac{5}{x-3} = \frac{-25}{x^2-x-6}$

$$x \neq -2, 3$$

$$\frac{3x \overset{(x+2)(x-3)}{}}{x+2} - \frac{5 \overset{(x+2)(x-3)}{}}{x-3} = \frac{-25 \overset{(x+2)(x-3)}{}}{(x-3)(x+2)}$$

$$3x(x-3) - 5(x+2) = -25$$

$$3x^2 - 9x - 5x - 10 = -25$$

$$3x^2 - 14x + 15 = 0 \quad \begin{array}{l} \times 45 \\ + -14 \\ -9, -5 \end{array}$$

$$\underline{3x^2 - 9x - 5x + 15 = 0}$$

$$3x(x-3) - 5(x-3) = 0$$

$$(x-3)(3x-5) = 0$$

$$x = \frac{5}{3}, \cancel{3}$$

$$\textcircled{x = \frac{5}{3}}$$

## 5.5 – Applications of Quadratic Equations

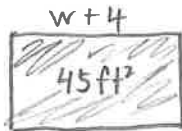
Here is some helpful vocabulary for word problems:

|                |   |                                 |
|----------------|---|---------------------------------|
| sum (+)        | } | whole numbers (0, 1, 2, 3, ...) |
| difference (-) |   | integers - non-decimals         |
| product (x)    |   | -3, -2, -1, 0, 1, 2, 3, ...     |
| quotient (÷)   |   |                                 |

reciprocal  $x = \frac{x}{1}$  so reciprocal is  $\frac{1}{x}$

consecutive: in a row

Example 1 – The area of a regulation Ping Pong table is  $45 \text{ ft}^2$ . The length is 4 ft more than the width. What are the dimensions of the table?



Let  $w = \text{width}$

so length =  $w + 4$

$A = lw$

$$45 = w(w + 4)$$

$$w^2 + 4w - 45 = 0$$

$$w = 5 \text{ ft}$$

$$(w + 9)(w - 5) = 0$$

$$\text{length} = w + 4 = 5 + 4 = 9 \text{ ft.}$$

$$w = \cancel{-9}, 5$$

reject

The dimensions are 9 ft by 5 ft.

Example 2 – The sum of a number and twice its reciprocal is  $\frac{9}{2}$ . Find the number.

Let  $x = \text{the number}$

$$x + 2\left(\frac{1}{x}\right) = \frac{9}{2}$$

$$x + \frac{2}{x} = \frac{9}{2} \quad \boxed{x \neq 0}$$

$$\frac{x(2x)}{1} + \frac{2(2x)}{x} = \frac{9(2x)}{2}$$

$$2x^2 + 4 = 9x$$

$$2x^2 - 9x + 4 = 0$$

$$2x^2 - 8x - x + 4 = 0$$

$$2x(x - 4) - 1(x - 4) = 0$$

$$(x - 4)(2x - 1) = 0$$

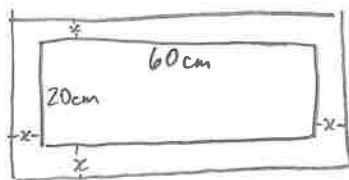
$$x = \frac{1}{2}, 4$$

There are two solutions:

$\frac{1}{2}$  and 4

Example 3 – A 20cm by 60cm painting has a frame surrounding it. If the frame is the same width all around, and the total area of the frame is  $516 \text{ cm}^2$ , how wide is the frame?

Let  $x = \text{width of frame}$



$$(2x + 60)(2x + 20) = 1716$$

$$4x^2 + 160x + 1200 = 1716$$

$$4x^2 + 160x - 516 = 0$$

$$\frac{4(x^2 + 40x - 129)}{4} = \frac{0}{4}$$

$$x^2 + 40x - 129 = 0$$

$$(x + 43)(x - 3) = 0$$

$$x = \cancel{-43}, 3$$

reject.

The width of the frame is 3cm.

$$A_{\text{frame only}} = 516 \text{ cm}^2$$

$$A_{\text{painting only}} = (60)(20) = 1200 \text{ cm}^2$$

$$A_{\text{painting + frame}} = 1200 + 516 = 1716 \text{ cm}^2$$

$$\text{length} = 2x + 60$$

$$\text{width} = 2x + 20$$

Example 4 – Sally biked from Mt. Douglas to Stelly's, a distance of 25km, on two consecutive days. On Day 1, she rode 3km/h faster so her ride took 20 minutes less. Calculate her speed on both days and round to the nearest tenth. Let  $x$  = speed on Day 2



|       | d (km) | s (km/h) | time (h)         |
|-------|--------|----------|------------------|
| Day 1 | 25     | $x+3$    | $\frac{25}{x+3}$ |
| Day 2 | 25     | $x$      | $\frac{25}{x}$   |

$$75x + x^2 + 3x = 75x + 225$$

$$x^2 + 3x - 225 = 0$$

cannot factor, so use quadratic formula.

$$a=1, b=3, c=-225$$

$$\frac{25}{x+3} + \frac{1}{3} = \frac{25}{x} \quad \boxed{x \neq -3, 0}$$

$$\frac{25}{x+3} + \frac{1}{3} = \frac{25}{x} \quad \begin{matrix} 3x(x+3) \\ 3x(x+3) \end{matrix}$$

$$75x + x(x+3) = 75(x+3)$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-225)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{909}}{2}$$

$x = 13.6, -16.6$   
 Day 2 speed  $\uparrow$  reject

Day 1 speed  
 $= x+3 = 13.6+3 = 16.6$

Sally's speed on Day 1 was 16.6 km/h and on Day 2 was 13.6 km/h.

Example 5 – The cold water tap can fill a container two hours faster than the hot water tap. The two taps together can fill the container in 80 minutes. How long does it take each tap to fill the container on its own? \*choose to work in hours or minutes.

Let  $x$  = time for just cold (mins)

$$\frac{\text{time together}}{\text{time alone 1}} + \frac{\text{time together}}{\text{time alone 2}} = 1$$

$$\frac{80}{x} + \frac{80}{x+120} = 1 \quad \boxed{x \neq 0, -120}$$

$$\frac{80}{x} + \frac{80}{x+120} = 1 \quad \begin{matrix} 80x(x+120) \\ 80x(x+120) \end{matrix}$$

$$80(x+120) + 80x = x(x+120)$$

$$80x + 9600 + 80x = x^2 + 120x$$

$$x^2 - 40x - 9600 = 0$$

$$(x-120)(x+80) = 0$$

$$x = 120, -80 \text{ reject}$$

$x = \text{cold only} = 120 \text{ min} = 2 \text{ hrs.}$

hot =  $x+120 = 120+120 = 240 \text{ min} = 4 \text{ hrs.}$

On their own, the cold tap takes 2h to fill the container and the hot tap takes 4h.

