3.1/3.3 – Investigating Quadratic Functions in Standard Form: $y = a(x \pm h)^2 \pm k$

Graph $y = x^2$ using a table of values

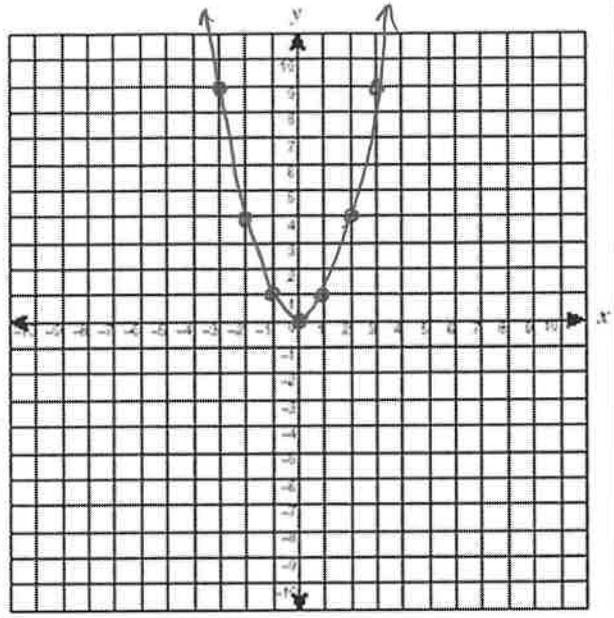
$y = x^2$

$y=(-3)^2=9$

$y=(-2)^2=4$

etc.

x	y	
-3	9	(-3,9)
-2	4	(-2,4)
-1	1	etc
0	0	
1	1	
2	4	
3	9	



Quick way to graph:

Use a basic count:

Start at vertex: in this case (0,0)

Over 1, up 1
back to vertex
Over 2, up 4
back to vertex
Over 3, up 9

Graph Shape: the graph shape is called a parabola and occurs when the equation has degree 2.

Parabolas have a vertex, a middle point. For $y = x^2$, it is (0, 0)

Parabolas have an AXIS OF SYMMETRY, a reflection line that splits the parabola into two symmetrical branches. It can be shown with a dashed line.

In this example, the equation of the axis of symmetry is $x = 0$

Parabolas open upward or downward. If they open upwards, they go up forever and ever, but only go down so far. Therefore, they have a minimum value. In the example above, the minimum value is $y = 0$. If they open downwards, they go down forever, but only go up so far. Therefore, they have a maximum value.

For any graph, you can find the domain. How far left does the graph go? How far right? In this example,

$x \in \mathbb{R}$

belongs to

the set of real numbers

For any graph, you can find the range. How far up does the graph go? How far down? In this example,

$y \geq 0$

A quadratic function is a function that has a second degree polynomial (has an x^2 term, but nothing higher. The graph shape that results is a PARABOLA.

Examples: $y = x^2 + 4$, $y = x^2 - 2x + 5$, $f(x) = (x+6)^2 - 4$

*Note: $f(x)$ is the same as y

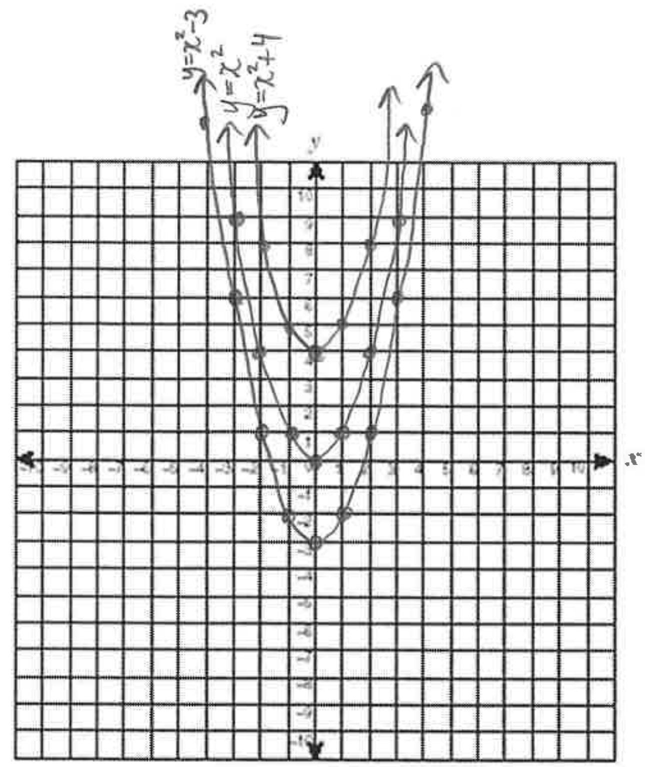
k value

$y = x^2 \pm k$

a) Graph $y = x^2$ using the basic count:
Start at (0,0) and go over 1, up 1

over 2, up 4

over 3, up 9



b) Graph $y = x^2 + 4$ using a table of values:

x	y
-3	13
-2	8
-1	5
0	4
1	5
2	8
3	13

$y = (-3)^2 + 4 = 9 + 4 = 13$
etc..

Notice: $y = x^2 + 4$ is the same parabola as $y = x^2$, only translated (moved) up 4 units. This is due to the 'k' value of 4 in the function.

Vertex: $(0, 4)$

A of S eqn: $x = 0$

Max/Min: $y = 4$

Domain: $x \in \mathbb{R}$

Range: $y \geq 4$

c) Graph $y = x^2 - 3$ by count method:

k value is: -3

Vertex is: $(0, -3)$

Then do basic count: over 1 up 1
2 4
3 9
4 16

Vertex: $(0, -3)$

A of S eqn: $x = 0$

Max/Min: $y = -3$

Domain: $x \in \mathbb{R}$

Range: $y \geq -3$

$y = x^2 \pm k$

The k value: translates the parabola vertically
- up if k is positive
- down if k is negative.

h value

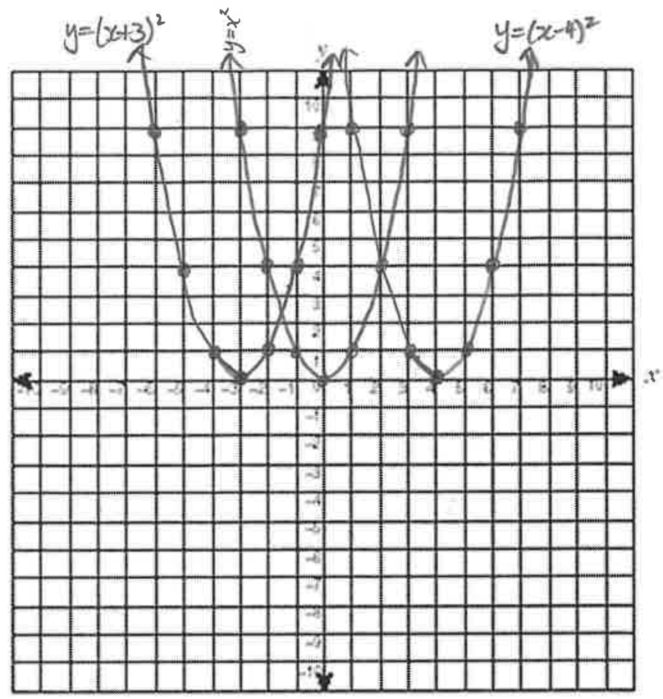
$y = (x \pm h)^2$

a) Graph $y = x^2$ using the count

b) Graph $y = (x-4)^2$ using a table of values

x	y
1	9
2	4
3	1
4	0
5	1
6	4
7	9

$y = (1-4)^2$
$= (-3)^2$
$= 9$
etc



Notice: $y=(x-4)^2$ is the same parabola as $y=x^2$ except it's translated 4 units right. This is due to the 'h' value of 4.

Vertex: $(4, 0)$

A of S eqn: $x = 4$

Max/Min: $y = 0$

Domain: $x \in \mathbb{R}$

Range: $y \geq 0$

h value Mental Switch: switch the sign of the constant in the brackets to get the 'h' value

c) Graph $y = (x+3)^2$ using the count method:

Vertex: $(-3, 0)$

A of S eqn: $x = -3$

Max/Min: $y = 0$

Domain: $x \in \mathbb{R}$

Range: $y \geq 0$

$y = (x \pm h)^2 \pm k$:

horizontal translation (switch sign)

vertical translation (it is what it is)

Vertex Notes: (h, k)

after sign switch

Practice

Ex – (a) Graph $y = (x+2)^2 - 5$ using the count method

Vertex: $(-2, -5)$

A of S eqn: $x = -2$

Max/Min: $y = -5$

Domain: $x \in \mathbb{R}$

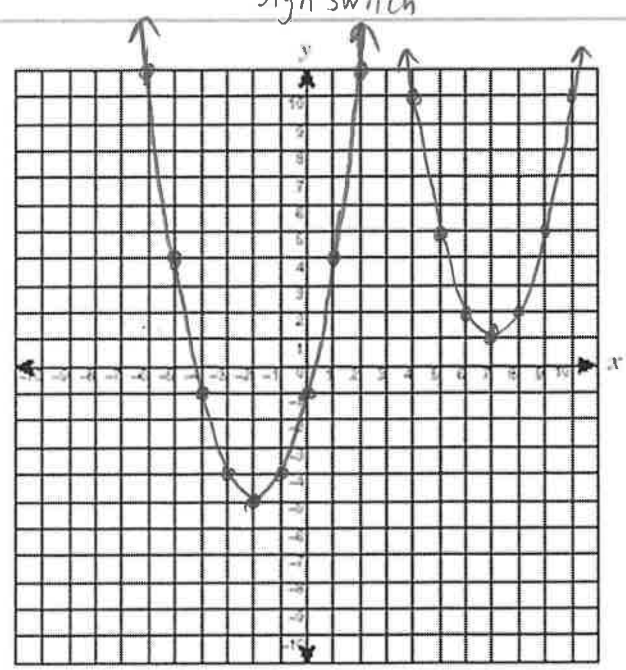
Range: $y \geq -5$

over	up
1	1
2	4
3	9
4	16

(b) Graph $y = (x-7)^2 + 1$

vertex $(7, 1)$

over	up
1	1
2	4
3	9



3.2/3.4 – Investigating Quadratic Functions in Standard Form: $y = a(x \pm h)^2 \pm k$

***a* value**

$y = ax^2$

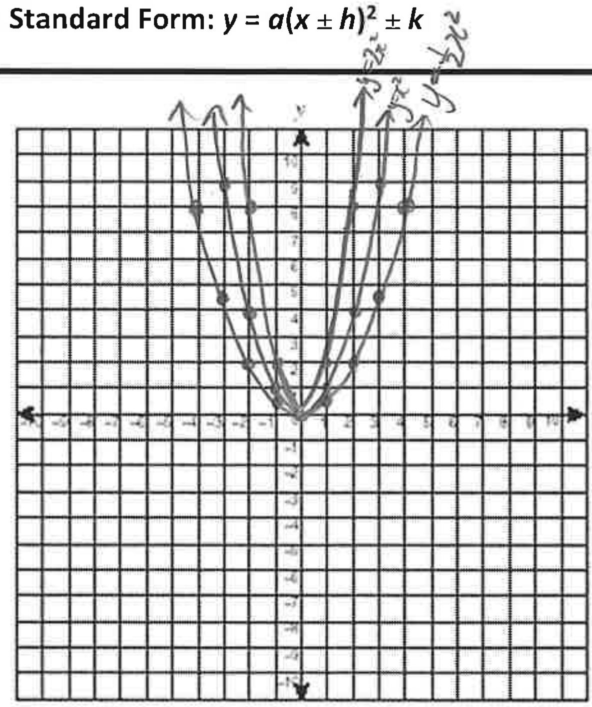
a)Graph $y = x^2$ using the count.

b) Graph $y = 2x^2$ using a table of values

x	y
-3	18
-2	8
-1	2
0	0
1	2
2	8
3	18

$y = 2(-3)^2$
$= 2(9)$
$= 18$

Notice: the parabola $y=2x^2$ is narrower (it rises faster - a vertical expansion) compared to $y=x^2$. This is due to the 'a' value of 2, which doubles the y values compared to $y=x^2$



The *a* value: alters the up/down count by the multiple 'a'.

c) Graph $y = \frac{1}{2}x^2$ using the count method:

vertex $(0,0)$ $a = \frac{1}{2}$

over	up
1	0.5
2	2
3	4.5
4	8

Graph $y = -x^2$ using a table of values

x	y
-3	-9
-2	-4
-1	-1
0	0
1	-1
2	-4
3	-9

$y = -(-3)^2$
$= -9$

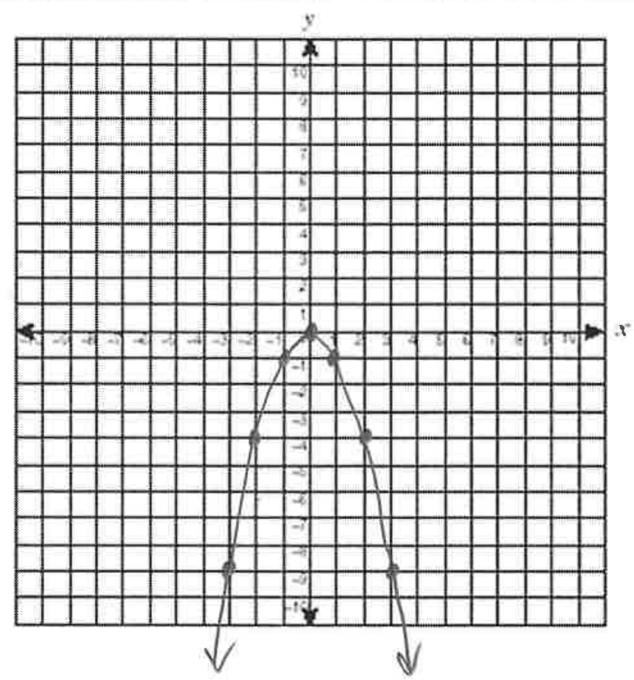
Vertex: $(0,0)$

Domain: $x \in \mathbb{R}$

A of S eqn: $x=0$

Range: $y \leq 0$

Max/Min: $y=0$

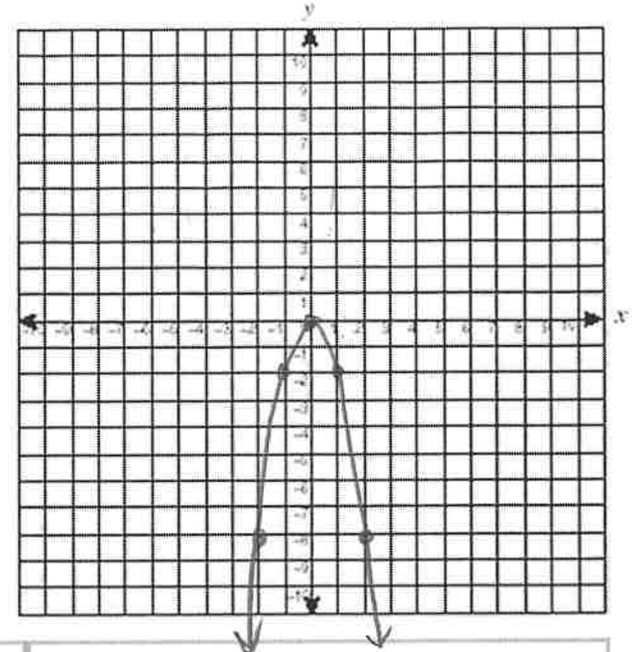


The $-a$ value: changes 'up' count to 'down' count

Graph $y = -2x^2$ using the count method

Vertex $(0,0)$

over 1 down 2
2 8
3 18



Standard Form:

$y = \pm a(x-h)^2 + k$

(+) up (−) down | alters up/down count | horizontal translation (sign switch) | vertical translation

Notes:

vertex (h,k)
axis of sym eqn $x=h$
Max/min $y=k$

Graph a)

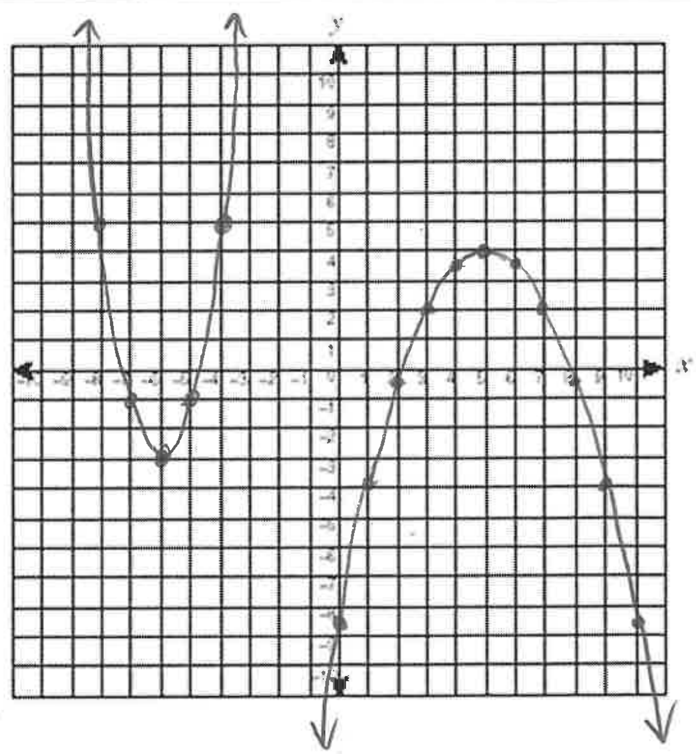
$f(x) = 2(x+6)^2 - 3$

and b) $y - 4 = -\frac{1}{2}(x-5)^2$

$y = -\frac{1}{2}(x-5)^2 + 4$

For each, find the

- vertex
- axis of sym eqn
- max/min
- domain
- range



a)Vertex: $(-6,-3)$

A of S eqn: $x=-6$

Max/Min: $y=-3$

Domain: $x \in \mathbb{R}$

Range: $y \geq -3$

b) Vertex: $(5,4)$

A of S eqn: $x=5$

Max/Min: $y=4$

Domain: $x \in \mathbb{R}$

Range: $y \leq 4$

x-ints

Thinking back to last chapter, what are x-intercepts?

where your graph touches/crosses the x axis

How many x-intercepts for a quadratic function?

0, 1, or 2

What are the methods we learned to identify x-intercepts?

factoring, quadratic formula, complete the square

Example – Determine the number of x-intercepts for each quadratic function, and also determine the y-intercept of each.

a) $y = -2(x-7)^2 - 1$

vertex $(7,-1)$

below x axis

$a=-2$ (opens down)

therefore, NO x-ints

b) $y = 0.5x^2 - 6$

vertex $(0,-6)$

below x axis

$a = 0.5$ (opens up)

therefore, TWO x-ints.

c) $y = -2(x+1)^2$

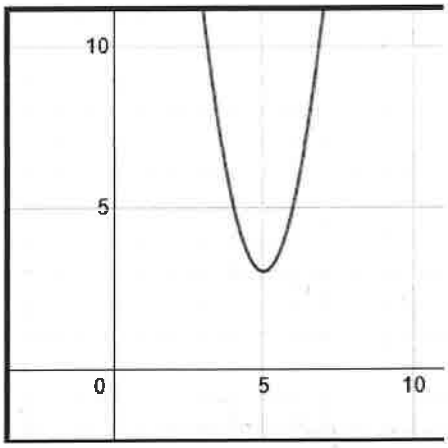
vertex $(-1,0)$

ON x axis

therefore, ONE x-int.

3.5 – Finding the Equation of a Parabola

Example 1 – Determine the equation of the following parabola:



vertex at $(5, 3)$

over 1, up 2 so $a = 2$

$y = 2(x-5)^2 + 3$

Example 2 – A parabola with vertex (1, -2) passes through the point (4, 1). Find the equation.

$y = a(x-h)^2 + k$

vertex $(1, -2)$

$y = a(x-1)^2 - 2$

sub $(4,1)$ in for x, y to solve for 'a'

$1 = a(4-1)^2 - 2$

$1 = a(3)^2 - 2$

$1 = 9a - 2$

$3 = 9a$

$a = \frac{3}{9} = \frac{1}{3}$

$y = \frac{1}{3}(x-1)^2 - 2$

Example 3 – Find the equation of a quadratic function whose graph has vertex (4, 8) and an *x*-intercept of 6.

$y = a(x-h)^2 + k$

$y = a(x-4)^2 + 8$

x-int of 6 is $(6, 0)$

$x \quad y$

$0 = a(6-4)^2 + 8$

$0 = 4a + 8$

$4a = -8$

$a = -2$

$y = -2(x-4)^2 + 8$

Example 4 –Write a quadratic function with a maximum of 3, axis of symmetry equation *x= -1*, that passes through (1, 1).

vertex $(-1, 3)$

$y = a(x+1)^2 + 3$

$1 = a(1+1)^2 + 3$

$1 = 4a + 3$

$-2 = 4a$

$a = \frac{-2}{4} = \frac{-1}{2}$

$y = -\frac{1}{2}(x+1)^2 + 3$

**Enrichment:* Find an equation of a quadratic function with points (3, -4), (-3, 2), & (1, 2).

$(-3,2)$ and $(1,2)$ are at same height (both have y of 2) so x value of vertex halfway between -3 and 1, which is -1

$y = a(x+1)^2 + k$

sub $(3,-4)$ into eqn

$-4 = a(3+1)^2 + k$

$-4 = 16a + k$

$k = -16a - 4$

so ...

$y = a(x-1)^2 - 16a - 4$

Sub in $(1,2)$

$2 = a(1+1)^2 - 16a - 4$

$2 = 4a - 16a - 4$

$6 = -12a$

$a = -\frac{1}{2}$

$y = -\frac{1}{2}(x+1)^2 + k$

$k = -16a - 4$

$k = -16(-\frac{1}{2}) - 4$

$k = 8 - 4$

$k = 4$

$y = -\frac{1}{2}(x+1)^2 + 4$

4.1 – Completing the Square

completing the square

When quadratic functions are in GENERAL FORM [$y = ax^2 \pm bx \pm c$], they can be changed into STANDARD FORM [$y = (x \pm h)^2 \pm k$] using a technique called completing the square.

Example 1 - Rewrite $y = 14 + 10x + x^2$ in standard form by completing the square. Then sketch the graph. Calculate the x-intercepts.

STEPS:

1) Rearrange so squared term is first and x term is second.
2) Find the *a, b, c* values
3) Take half the b-value (you'll need this later), then square it.
4) Add and subtract the result to your quadratic function after the *x* term.
5) Make sure the new term you added is the third term.
6) Factor the trinomial and add the two last terms.

Shortcut for factoring the trinomial:

the 'magic numbers' are just the halved 'b' value

$y = x^2 + 10x + 14$

$b = 10, 5, 25$

$y = x^2 + 10x + 25 - 25 + 14$

$y = (x^2 + 10x + 25) - 25 + 14$

$y = (x+5)^2 - 11$

vertex $(-5, -11)$

$a = 1$

over 1 up 1

2 4

3 9

4 16

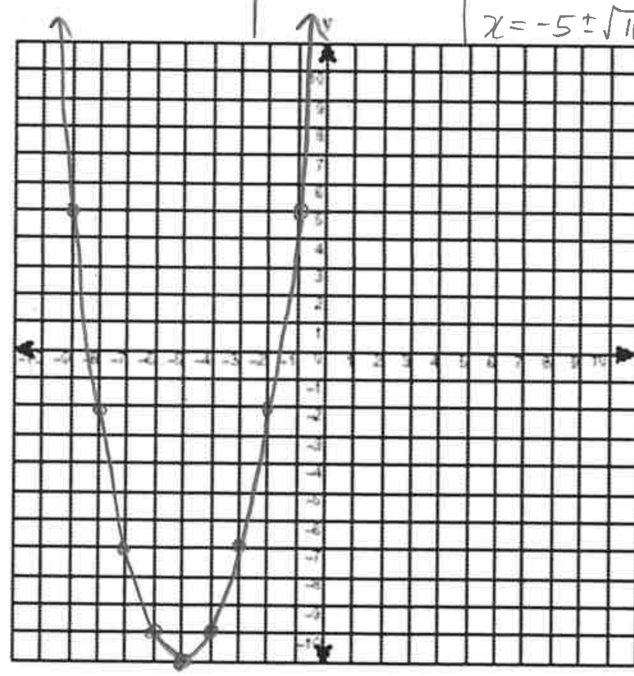
x-ints:

$0 = (x+5)^2 - 11$

$11 = (x+5)^2$

$x + 5 = \pm\sqrt{11}$

$x = -5 \pm \sqrt{11}$



Example 2- Change $y = x^2 - 4x - 1$ into standard form, then calculate the x-intercepts.

$b = -4, -2, 4$

$y = (x^2 - 4x + 4) - 4 - 1$

$y = (x-2)^2 - 5$

vertex $(2, -5)$

x-ints

$0 = (x-2)^2 - 5$

$(x-2)^2 = 5$

$x - 2 = \pm\sqrt{5}$

$x = 2 \pm \sqrt{5}$

When a ≠ 1

When the *a* value is different from 1, there are a few more steps.

Example 3 - Change $y = -2x^2 + 4x + 5$ into standard form and then find *x*-ints

STEPS:

1) Group the first two terms together.
2) Factor the *a* value out.
3) Find the *b* value. Take half and square it.
4) Add and subtract the result IN THE BRACKETS.
5) Get the subtracted result out of the brackets by multiplying to the coefficient in front of the brackets.
6) Factor the trinomial.

$y=(-2x^2+4x)+5$

$y=-2(x^2-2x)+5$

$b=-2, -1, 1$

$y=-2(x^2-2x+1-1)+5$

$y=-2(x^2-2x+1)+2+5$

$y=-2(x-1)^2+7$

Vertex $(1,7)$

x-ints

$0=-2(x-1)^2+7$

$-7=-2(x-1)^2$

$\frac{7}{2}=(x-1)^2$

$x-1=\pm\frac{\sqrt{7}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$

$x-1=\pm\frac{\sqrt{14}}{2}$

$x=1\pm\frac{\sqrt{14}}{2}$

$x=\frac{2\pm\sqrt{14}}{2}$

Example 4 - Change $y = 3x^2 - 12x + 11$ into standard form, then calculate the *x*-ints.

$y=(3x^2-12x)+11$

$y=3(x^2-4x)+11$

$b=-4, -2, 4$

$y=3(x^2-4x+4-4)+11$

$y=3(x^2-4x+4)-12+11$

$y=3(x-2)^2-1$

x-ints

$0=3(x-2)^2-1$

$1=3(x-2)^2$

$(x-2)^2=\frac{1}{3}$

$x-2=\pm\frac{\sqrt{1}}{\sqrt{3}}$

$x-2=\pm\frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}$

$x-2=\pm\frac{\sqrt{3}}{3}$

$x=2\pm\frac{\sqrt{3}}{3}$

$x=\frac{6\pm\sqrt{3}}{3}$

Example 5 - Change $y = 5x - 3x^2 + 1$ into standard form using exact values.

$y=(-3x^2+5x)+1$

$y=-3(x^2-\frac{5}{3}x)+1$

$b=-\frac{5}{3}, -\frac{5}{6}, \frac{25}{36}$

$y=-3(x^2-\frac{5}{3}x+\frac{25}{36}-\frac{25}{36})+1$

$y=-3(x^2-\frac{5}{3}x+\frac{25}{36})+\frac{75}{36}+\frac{36}{36}$

$y=-3(x-\frac{5}{6})^2+\frac{111\div 3}{36\div 3}$

$y=-3(x-\frac{5}{6})^2+\frac{37}{12}$

Vertex $(\frac{5}{6}, \frac{37}{12})$

4.3 – Applications of Quadratic Functions

Example 1 - The path of a rocket fired over a lake is described by the function

$h(t) = -4.9t^2 + 49t + 1.5$ where *h(t)* is the height of the rocket, in metres, and *t* is time in seconds, since the rocket was fired.

a) What is the maximum height reached by the rocket? How many seconds after it was fired did the rocket reach this height?
b) How high was the rocket above the lake when it was fired?
c) At what time does the rocket hit the ground?
d) What domain and range are appropriate in this situation?
e) How high was the rocket after 7s? Was it on its way up or down?

a) max is at vertex so complete the square to get the vertex

$h(t) = (-4.9t^2 + 49t) + 1.5$

$h(t) = -4.9(t^2 - 10t) + 1.5$

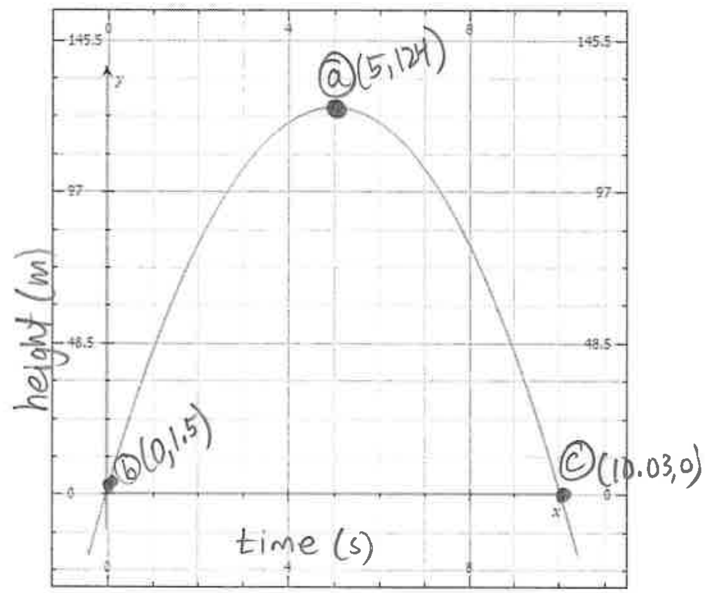
$b = -10, -5, 25$

$h(t) = -4.9(t^2 - 10t + 25 - 25) + 1.5$

$h(t) = -4.9(t^2 - 10t + 25) + 122.5 + 1.5$

$h(t) = -4.9(t - 5)^2 + 124$

vertex $(5, 124)$ The max height of the rocket was 124 m, reach 5s after launch.



b) fired at $t=0$.

$h(0) = -4.9(0)^2 + 49(0) + 1.5$

$h(0) = 1.5$ It was 1.5m above the lake when fired.

c) $h = 0$ when rocket hits ground

$0 = -4.9(t-5)^2 + 124$

$-124 = -4.9(t-5)^2$

$25.306 = (t-5)^2$

$t - 5 = \pm\sqrt{25.306}$

$t - 5 = \pm 5.03$

$t = 10.03, \cancel{-0.03}$ reject

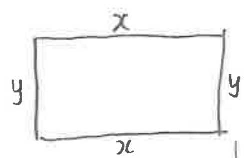
The rocket hit the ground 10.03s after launch

***Keep in mind that the question presented this function in general form. Sometimes, in problems like this, the function is presented in standard form, which will make it much easier**

Max/Min Problems:

Example 2 – At a concert, organizers are roping off a rectangular area for sound equipment. There is 160m of fencing available to create the perimeter. What dimensions will give the maximum area, and what is the maximum area?

Steps:
1) Write an equation for perimeter, and write an equation for area for a rectangle.
2) Use the two equations to create a quadratic function in general form.
3) Complete the square to change the quadratic function into standard form.
4) Identify the maximum area, and then the dimensions for the maximum area.



$P = 2x + 2y$

① $160 = 2x + 2y$

② $A = xy$

$\frac{2y}{2} = \frac{160}{2} - \frac{2x}{2}$

$y = 80 - x$

$A = x(80-x)$

$A = -x^2 + 80x$

complete the square to get the vertex, which is the max. area.

$A = -(x^2 - 80x)$

$b = -80, -40, 1600$

$A = -(x^2 - 80x + 1600 - 1600)$

$A = -(x-40)^2 + 1600$

vertex $(40, 1600)$

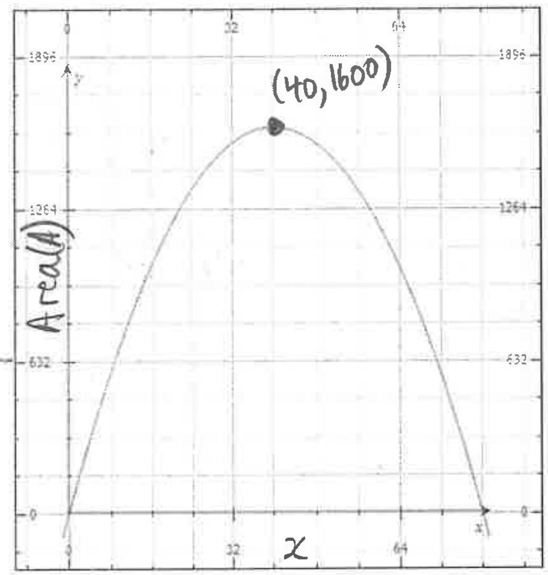
x ↑ ↑ A

If $x = 40$, what is y?

$160 = 2(40) + 2y$

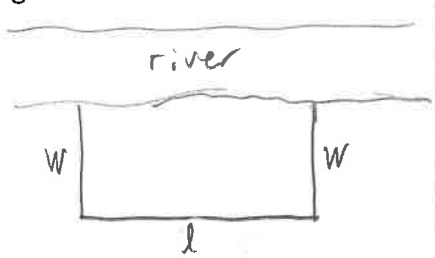
$2y = 80$

$y = 40$



Dimensions of 40m by 40m allow for a maximum area of $1600m^2$.

Example 3 – A rancher has 800m of fencing to enclose a rectangular cattle pen along a river bank. There is no fencing needed along the river bank. Find the dimensions that would enclose the largest area.



$800 = 2w + l$

$A = lw$

$l = 800 - 2w$

$A = w(800 - 2w)$

$A = -2w^2 + 800w$

$A = -2(w^2 - 400w)$

$b = -400, -200, 40000$

$A = -2(w^2 - 400w + 40000 - 40000)$

$A = -2(w^2 - 400w + 40000) + 80\,000$

$A = -2(w - 200)^2 + 80\,000$

vertex $(200, 80\,000)$

w width ↑ ↑ max A Area

Find l:

$800 = 2(200) + l$

$800 = 400 + l$

$l = 400$

The dimensions that will enclose a max area of $80\,000 m^2$ are 400m by 200m

Example 4 – A sporting goods store sells basketball shorts for $8. At this price their weekly sales are approximately 100 items. Research says that for every $2 increase in price, the manager can expect the store to sell five fewer pairs of shorts. Determine the maximum revenue the manager can expect based on these estimates. What selling price will give that maximum revenue, and how many shorts will be sold?

Let $x =$ the number of $2 increases in price

Revenue = (Items Sold)(Price)

Price $= 8+2x$

Items Sold $= 100-5x$

$$R = (8+2x)(100-5x)$$

$$R = -10x^2+160x+800$$

$$R = (-10x^2+160x)+800$$

$$R = -10(x^2-16x)+800$$

$b = -16, -8, 64$

$$R = -10(x^2-16x+64-64)+800$$

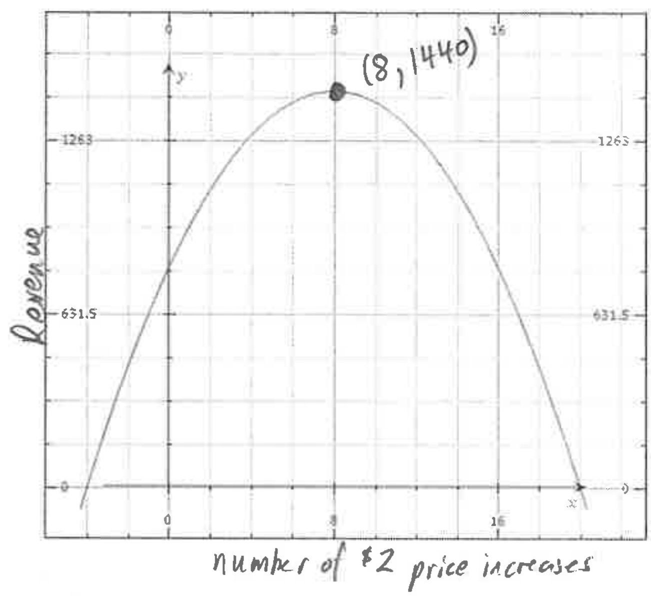
$$R = -10(x^2-16x+64)+640+800$$

$$R = -10(x-8)^2+1440$$

vertex (8, 1440)

x (the number of $2 price increases)

max revenue



A price of $24 per pair will cause 60 pairs of shorts to be sold per week, resulting in a max revenue of $1440.

Price $= 8+2x = 8+2(8)$

$= \$24$

Items Sold $= 100-5x$

$100-5(8)$

60