

3) If n is odd, then there is ONE real n^{th} root of a .

Example 4 – Solve for x

a) $x^3 = 8$

b) $x^3 = -8$

c) $x^5 = -4$

4) If a is zero, then there is ONE real n^{th} root of a , and it is ZERO.

Example 5 – Solve for x

a) $x^2 = 0$

b) $x^5 = 0$

Radical Properties from Math 10:

1) $a^{\frac{1}{n}} = \sqrt[n]{a}$ as discussed in above notes

2) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ Example: $16^{\frac{3}{4}}$

3) $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{(\sqrt[n]{a})^m}$ Example: $27^{-\frac{2}{3}}$

4) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ Example: $\sqrt[3]{\frac{27}{64}}$

5) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ Example: $\sqrt{12}$

1.3B – Radical Operations

absolute value

Absolute Value is “how many jumps the number is from zero”. Stated another way, it is the distance from zero on the number line, regardless of direction. Distances are always POSITIVE values. 3 is 3 jumps from zero, so the absolute value of 3, or $|3| = \underline{\hspace{1cm}}$. -3 is 3 jumps from zero, so $|-3| = \underline{\hspace{1cm}}$. So if $|x| = 3$, x could have been $\underline{\hspace{1cm}}$ OR $\underline{\hspace{1cm}}$. For every absolute value solution, there is a positive and negative possibility.

Example 1 - Evaluate

a) $|5|$ b) $|-7|$ c) $|-0.34|$ d) $|\frac{5}{6}|$ e) $|-6\frac{3}{8}|$

Example 2 - What are the possible values of x ?

a) $|x| = 6$ b) $|x| = 9.7$ c) $|x| = -2$

Radical Review

Example 3 - Identify and define all parts of the radical, then simplify:

$$5\sqrt[3]{8}$$

Roots of Positive Powers of x :

Case 1: When $x \geq 0$ in $\sqrt{x^n}$ with n a positive integer.

The square roots of negative numbers are undefined in the set of real numbers. Therefore, if $x \geq 0$, simplification is easier to realize.

For example:

Is x^2 a perfect square?

$$\sqrt{x^2} =$$

Is x^2 a perfect cube? etc.

$$\sqrt[3]{x^2} =$$

Is x^3 a perfect sq?

Does it have any factors that are?

$$\sqrt{x^3} =$$

$$\sqrt[3]{x^3} =$$

$$\sqrt{x^4} =$$

$$\sqrt[3]{x^4} =$$

$$\sqrt{x^5} =$$

$$\sqrt[3]{x^5} =$$

$$\sqrt{x^6} =$$

$$\sqrt[3]{x^6} =$$

Example 4 – Simplify. Assume all variables represent positive numbers

a) $\sqrt{16y^2}$ b) $\sqrt{x^4y^3}$ c) $\sqrt{25x^5y^3z^2}$ d) $\sqrt[3]{8x^4y^5}$ e) $\sqrt[3]{27x^3y^6}$

Before Case 2 is discussed, it's important to understand the **Principal Square Root Theorem**. Every positive number 'n' has two square roots. One is positive, and the other is negative.

Example: $x^2 = 16$, so $x = \pm\sqrt{16} = \pm 4$

The **PRINCIPAL SQUARE ROOT** is the **POSITIVE NUMBER SQUARE ROOT**.

Unless otherwise stated, the 'square root' of a number refers **ONLY** to the principal square root. *A general rule is that if the radical is present originally in the question (rather than you introducing it to the question), give only the principal square root as the answer.

Thus, $\sqrt{16} = 4$ but $x^2 = 16$
 $x = \pm\sqrt{16} = \pm 4$

Case 2: When x is a real number (meaning it could be positive, zero, or negative – we don't know which at first) in $\sqrt{x^n}$, with n an even integer, then $\sqrt{x^2} = x$ like we determined for case 1 (when x couldn't be negative) is no longer sufficient. Here is why:

Let's say that we know that x is negative, for example $x = -3$.

$$\sqrt{(-3)^2} =$$

How do we 'fix' this issue?

When variables exist in a radicand, it is not known if the variable represents a negative number, zero, or a positive number. An **ABSOLUTE VALUE** is sometimes needed to ensure that the result is a positive number (so that your solution will be the principal square root). If x can be any real number (meaning it *could* be negative), then:

$$\sqrt{x^2} = |x|$$

This will allow for x to not magically change, but also allow for the principal square root (the positive root) to be the one and only solution.

Let's say that we know that x is negative, for example $x = -3$.

Then:

This is only necessary when x changes from an **EVEN** power to an **ODD** power.

$$\sqrt{x^4} = x^2 \text{ (no absolute value needed as } x^2 \text{ can only result as a positive)}$$

For example, if $x = -3$:

$$\text{And moving forward: } \sqrt{x^6} = \sqrt{x^8} = \sqrt{x^{10}} = \text{etc.}$$

If the exponent in the radicand is **ODD**, then a **NEGATIVE** value of x will make the value **NEGATIVE**, which is **UNDEFINED** in the real number system. Therefore, $x \geq 0$ for all odd exponents. Examples: $\sqrt{x^3} = \sqrt{x^2 x} = x\sqrt{x}; x \geq 0$

$$\sqrt{x^5} = \sqrt{x^7} =$$

Example 5 – Simplify. Let the variables be any real numbers.

$$\text{a) } \sqrt{16x^2} \quad \text{b) } \sqrt{25x^2y^4} \quad \text{c) } \sqrt{x^{14}} \quad \text{d) } \sqrt{36x^2y^5} \quad \text{e) } \sqrt{32x^6y^3z^8}$$

Summary:

1.4 – Simplifying Radicals

Radicals can be written as fractional exponents, as learned in Math 10.

Examples: $\sqrt{2} = 2^{\frac{1}{2}}$ $\sqrt[3]{x} = x^{\frac{1}{3}}$ Generally: $\sqrt[n]{a} = a^{\frac{1}{n}}$

Three Important Relationships of Radicals, all from Math 10:

1) $\sqrt[n]{a^n} = a, a \geq 0$ because $\sqrt[n]{a^n} = a^{\frac{n}{n}} = a^1$ Example: $\sqrt[2]{5^2} =$

2) $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}, a, b \geq 0$ because $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = \sqrt[n]{a} \times \sqrt[n]{b}$

3) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, a \geq 0, b > 0$ because $\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Simplifying Expressions Containing Radicals

Example 1 – Simplify $\sqrt{20}$

Two Methods:

It is beneficial to know the perfect squares up to 144, perfect cubes up to 125, and perfect fourths up to 81.

Example 2 – Simplify

a) $\sqrt{8}$

b) $\sqrt{27}$

c) $3\sqrt{52}$

d) $\sqrt[3]{24}$

e) $5\sqrt[3]{81}$

f) $\sqrt[4]{32}$

Example 3 – Simplify (assume variables are positive)

a) $\sqrt{18x^3y^6}$

b) $\sqrt{63n^7p^4}$

c) $\sqrt{32x^8y^{11}}$

d) $\sqrt[3]{40a^4b^8c^{15}}$

e) $\sqrt[3]{54a^5b^{10}}$

f) $\sqrt[4]{m^7}$

g) $\sqrt[4]{162x^3y^{11}z^5}$

h) $\sqrt[3]{\frac{x^{13}}{64}}$

Changing Mixed Radicals to Entire Radicals

Example 4 – Change to Entire (assume variables are positive)

a) $4\sqrt{3}$

b) $3\sqrt{5}$

c) $2\sqrt[3]{7}$

d) $-2x\sqrt{6x}$

e) $x^3\sqrt{x}$

f) $3a^2b\sqrt[3]{b^2c}$

g) $\frac{3x^2y}{5}\sqrt[3]{2xy^2}$

1.5 – Adding and Subtracting Radical Expressions

Like Radicals

'Like Radicals' work very similar to 'Like Terms'.

Simplify: $3x + 2x$

Simplify $3\sqrt{2} + 2\sqrt{2}$

Like radicals have _____

Steps for adding & subtracting like radicals:

Example 1 - Simplify

a) $7\sqrt{3} - 2\sqrt{3}$

b) $-5\sqrt[3]{10} - 6\sqrt[3]{10}$

c) $4\sqrt{2} - 5\sqrt[3]{2}$

d) $2\sqrt{75} + 3\sqrt{3}$

e) $-\sqrt{27} + 3\sqrt{5} - \sqrt{80} - 2\sqrt{12}$

f) $\sqrt{9b} - 3\sqrt{16b}$, $b \geq 0$

In example f, why does b have to be greater than or equal to zero?

Example 2 – Simplify. Assume variables are positive values.

a) $\sqrt{27xy} + \sqrt{8xy}$

b) $4\sqrt[3]{16} + 3\sqrt[3]{54}$

c) $3x\sqrt{63y} - 5\sqrt{28x^2y}$

d) $\frac{5}{2}\sqrt[3]{16x^4y^5} - xy\sqrt[3]{54xy^2}$

1.6A – Multiplying & Dividing Radical Expressions

**multiplying
radicals**

Example 1 - Multiply $2\sqrt{5} (3\sqrt{5})$ Verify your answer:

To multiply radicals:

In general:

Example 2 - Simplify: a) $5\sqrt{3} (\sqrt{6})$

b) $2\sqrt{6} (4\sqrt{8})$

c) $-3\sqrt{2x} (4\sqrt{3x})$ $x \geq 0$

d) $-2\sqrt[3]{11}(4\sqrt[3]{2} - 3\sqrt[3]{3})$

e) $(4\sqrt{2} + 3)(\sqrt{7} - 5\sqrt{14})$

f) $(\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2})$

**dividing
radicals**

Example 3 - Divide $\frac{6\sqrt{12}}{3\sqrt{6}}$

Verify your answer:

To divide radicals:

In general:

Example 4 - Simplify: a) $\frac{-24\sqrt[3]{14}}{8\sqrt[3]{2}}$ b) $\frac{2\sqrt{51}}{\sqrt{3}}$ c) $\frac{\sqrt{18x^3}}{\sqrt{3x}}, x > 0$

Multiplying & Dividing Terms with Different Indices

Example 5 – Simplify $\sqrt{x^3} (\sqrt[3]{x}), x \geq 0$

Example 6 – Simplify $\frac{\sqrt{x^3}}{\sqrt[3]{x}}, x > 0$

1.6B – Rationalizing the Denominator

rationalizing
the
denominator

A final answer cannot have a radical in the denominator. Therefore, you may have to '**rationalize the denominator**' – a process that will eliminate the radical from the denominator without changing the value of the expression.

Example 1 - Rationalize: a) $\frac{3}{\sqrt{5}}$ b) $\sqrt{\frac{2}{7}}$ c) $\frac{6}{7\sqrt{5}}$ d) $\frac{2\sqrt{5}}{3^3\sqrt{6}}$

If the denominator is a radical *monomial*, multiply the numerator and denominator by that radical.

e) $\sqrt[3]{\frac{2}{y}}$

f) $\frac{\sqrt[4]{5x}}{\sqrt[4]{10x^3}}$

g) $\frac{2}{\sqrt{x+1}}$

Example 2 – Rationalize: a) $\frac{3}{\sqrt{x}-2}$

b) $\frac{2+\sqrt{2}}{3\sqrt{5}-4}$

If the denominator is a radical *binomial*, multiply the numerator & denominator by its **conjugate**.

c) $\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}-\sqrt{2b}}$, $a, b \geq 0$

Example 3 - Simplify

a) $6\sqrt{\frac{3}{4x}}$, $x > 0$

b) $\frac{-7}{2^3\sqrt{9p}}$, $p \neq 0$

Example 4 – The surface area of a sphere is $S = 4\pi r^2$. If the surface area of the sphere is 144 mm^2 , what is the radius?

1.7 – Radical Equations

Radical Equations are mathematical equations that include a radical, such as:

$2\sqrt{6x} - 1 = 11$. If the index of the radical is even, there are restrictions on the variable: Since it is not possible to find the square root of a negative number, the radicand cannot be negative.

Example 1 – Find the restriction on the variable:

a) $2\sqrt{6x} - 1 = 11$

b) $\sqrt{x + 2} = 49$

c) $7\sqrt{-2x + 3} = 35$

d) $\sqrt{3x + 4} = \sqrt{2x - 4}$

Steps to solving radical equations:

1. Find the restrictions on the variable in the radicand (if the index is even). Remember, the radicand must be set to ≥ 0 and then solved (if you multiply or divide by a negative number to both sides, FLIP the inequality).
2. Get the radical all by itself on one side of the equation.
3. If the index is 2, square both sides (if index is 3, cube both sides, etc.) and then solve for the variable.
4. See if the solution is affected by the restriction.
5. Check the answer using the original equation to see if solutions are valid or extraneous.

Example 2 – Solve a) $2\sqrt{6x} - 1 = 11$

b) $-8 + \sqrt{\frac{3y}{5}} = 2$

Example 3 – Solve a) $4 + \sqrt{2x - 3} = 1$

b) $-2\sqrt{x - 5} = 16$

Example 4 - Solve

a) $\sqrt{10x - 7} = 3\sqrt{x}$

b) $2\sqrt{x} = \sqrt{7x + 6}$

Example 5 – Solve

a) $\sqrt{x + 1} = x - 1$

b) $m - \sqrt{2m + 3} = 6$

