Example 1 – Simplify

a) 4^2 b) $16^{\frac{1}{2}}$ c) $\sqrt{16}$ d) 2^3 e) $8^{\frac{1}{3}}$ f) $\sqrt[3]{8}$

index index

16 4 8 8 2 2

What is the relationship between a, b, c? What is the relationship between d, e, f?

'square rooting' is the opposite of 'squaring'! Exponent of $\frac{1}{2}$ is square rooting.

'cube rooting' is the opposite of 'cubing'! Exponent of $\frac{1}{3}$ is cube rooting.

RADICAL EXPRESSION

ex $2^3 = 8$ $2 = 8^{\frac{1}{3}} = \sqrt[3]{8}$

If $x^n = a$, then $x = a^{\frac{1}{n}} = \sqrt[n]{a}$

If $'a'$ and x are real numbers and n is a positive integer, then x is an n^{th} root of $'a'$ (ie $x = a^{\frac{1}{n}} = \sqrt[n]{a}$) if $x^n = a$

n^{th} root theorems: $x^n = a$

1) If a is positive and n is even, then there exist TWO real n^{th} roots.

Example 2 – Solve for x

a) $x^2 = 16$

$\sqrt{x^2} = \pm\sqrt{16}$

$x = \pm 4$

b/c

$(4)(4) = 16$

and

$(-4)(-4) = 16$

b) $x^2 = 11$

$x = \pm\sqrt{11}$

b/c

$(\sqrt{11})(\sqrt{11}) = 11$

and

$(-\sqrt{11})(-\sqrt{11}) = 11$

c) $x^4 = 81$

$x = \pm\sqrt[4]{81}$

$x = \pm 3$

d) $x^4 = 5$

$x = \pm\sqrt[4]{5}$

2) If a is negative and n is even, then there are NO real number solutions.

Example 3 – Solve for x

a) $x^2 = -25$

$x = \pm\sqrt{-25}$

$\emptyset$ cannot sq root a negative number

b) $x^4 = -7$

$x = \pm\sqrt[4]{-7}$

$\emptyset$

3) If n is odd, then there is ONE real n^{th} root of a.

Example 4 – Solve for x

a) $x^3 = 8$

$x = \sqrt[3]{8}$

$x = 2$

b) $x^3 = -8$

$x = \sqrt[3]{-8}$

$x = -2$

c) $x^5 = -4$

$x = \sqrt[5]{-4}$

4) If a is zero, then there is ONE real n^{th} root of a, and it is ZERO.

Example 5 – Solve for x

a) $x^2 = 0$

$x = \pm\sqrt{0}$

$x = 0$

b) $x^5 = 0$

$x = \sqrt[5]{0}$

$x = 0$

Radical Properties from Math 10:

1) $a^{\frac{1}{n}} = \sqrt[n]{a}$ as discussed in above notes

2) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ Example: $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8$

3) $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{(\sqrt[n]{a})^m}$ Example: $27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$

4) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ Example: $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$

5) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ Example: $\sqrt{12}$

$= \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$

1.3B – Radical Operations

absolute value

Absolute Value is "how many jumps the number is from zero". Stated another way, it is the distance from zero on the number line, regardless of direction. Distances are always POSITIVE values. 3 is 3 jumps from zero, so the absolute value of 3, or $|3| =$ 3. -3 is 3 jumps from zero, so $|-3| =$ 3. So if $|x| = 3$, x could have been 3 OR -3. For every absolute value solution, there is a positive and negative possibility.

Example 1 - Evaluate

a) $|5|$ — 5

b) $|-7|$ — 7

c) $|-0.34|$ — 0.34

d) $\left|\frac{5}{6}\right|$ — $\frac{5}{6}$

e) $\left|-6\frac{3}{8}\right|$ — $6\frac{3}{8}$

Example 2 - What are the possible values of x?

a) $|x| = 6$ — $x = \pm 6$

b) $|x| = 9.7$ — $x = \pm 9.7$

c) $|x| = -2$ — $\emptyset$

Radical Review

Example 3 - Identify and define all parts of the radical, then simplify:

index

coefficient $\rightarrow 5\sqrt[3]{8}$ radical sign (root)

radicand

Roots of Positive Powers of x:

Case 1: When $x \geq 0$ in $\sqrt{x^n}$ with n a positive integer.

The square roots of negative numbers are undefined in the set of real numbers. Therefore, if $x \geq 0$, simplification is easier to realize.

For example:

Is x^2 a perfect square? Yes.

$\sqrt{x^2} = x$

Is x^3 a perfect sq? No

Does it have any factors that are? Yes

$\sqrt{x^3} = \sqrt{x^2 x} = x\sqrt{x}$

$\sqrt{x^4} = x^2$

$\sqrt{x^5} = \sqrt{x^4 x} = x^2\sqrt{x}$

$\sqrt{x^6} = x^3$

$\sqrt{x^7} = \sqrt{x^6 x} = x^3\sqrt{x}$

Is x^2 a perfect cube? etc. No

$\sqrt[3]{x^2} = \sqrt[3]{x^2}$

$\sqrt[3]{x^3} = x$

$\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = x\sqrt[3]{x}$

$\sqrt[3]{x^5} = \sqrt[3]{x^3 x^2} = x\sqrt[3]{x^2}$

$\sqrt[3]{x^6} = x^2$

$\sqrt[3]{x^7} = \sqrt[3]{x^6 x} = x^2\sqrt[3]{x}$

Example 4 – Simplify. Assume all variables represent positive numbers

* I like to circle what stays in the radical

a) $\sqrt{16y^2}$

$4y$

b) $\sqrt{x^4y^3}$

$\sqrt{x^4y^2y}$

$x^2y\sqrt{y}$

c) $\sqrt{25x^5y^3z^2}$

$\sqrt{25x^4xy^2yz^2}$

$5x^2yz\sqrt{xy}$

d) $\sqrt[3]{8x^4y^5}$

$\sqrt[3]{8x^3xy^3y^2}$

$2xy\sqrt[3]{xy^2}$

e) $\sqrt[3]{27x^3y^6}$

$3xy^2$

Before Case 2 is discussed, it's important to understand the **Principal Square Root Theorem.**
Every positive number 'n' has two square roots. One is positive, and the other is negative.
Example: $x^2 = 16$, so $x = \pm\sqrt{16} = \pm 4$
*The **PRINCIPAL SQUARE ROOT** is the **POSITIVE NUMBER SQUARE ROOT**.*
*Unless otherwise stated, the 'square root' of a number refers **ONLY** to the principal square root. *A general rule is that if the radical is present originally in the question (rather than you introducing it to the question), give only the principal square root as the answer.*
Thus, $\sqrt{16} = 4$ but $x^2 = 16$

$$x = \pm\sqrt{16} = \pm 4$$

Case 2: When x is a real number (meaning it could be positive, zero, or negative – we don't know which at first) in $\sqrt{x^n}$, with n an even integer, then $\sqrt{x^2} = x$ like we determined for case 1 (when x couldn't be negative) is no longer sufficient. Here is why:
Let's say that we know that x is negative, for example $x = -3$.

$\sqrt{(-3)^2} = \sqrt{9} = 3$ If $\sqrt{x^2} = x$, the answer would be -3. But it cannot be, due to the principal sq root theorem.

How do we 'fix' this issue?
When variables exist in a radicand, it is not known if the variable represents a negative number, zero, or a positive number. An **ABSOLUTE VALUE** is sometimes needed to ensure that the result is a positive number (so that your solution will be the principal square root).
If x can be any real number (meaning it *could* be negative), then:

$$\sqrt{x^2} = |x|$$

This will allow for x to not magically change, but also allow for the principal square root (the positive root) to be the one and only solution.
Let's say that we know that x is negative, for example $x = -3$. Thus, x is still -3, but the absolute value gives us a positive answer to fulfill the principal sq root theorem.
Then: $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = |-3| = 3$

This is only necessary when x changes from an **EVEN** power to an **ODD** power.
$\sqrt{x^4} = x^2$ (no absolute value needed as x^2 can only result as a positive)
For example, if $x = -3$:

And moving forward: $\sqrt{x^6} = |x^3|$ $\sqrt{x^8} = x^4$ $\sqrt{x^{10}} = |x^5|$ etc.
If the exponent in the radicand is **ODD**, then a **NEGATIVE** value of x will make the value **NEGATIVE**, which is **UNDEFINED** in the real number system. Therefore, $x \geq 0$ for all odd exponents. Examples: $\sqrt{x^3} = \sqrt{x^2\,x} = x\sqrt{x}\,; x \geq 0$ $x\sqrt{x}$ is only the solution if $x \geq 0$. If x is negative, then $\varnothing$

$\sqrt{x^5} = \sqrt{x^4\,x} = x^2\sqrt{x},\ x \geq 0$ $\sqrt{x^7} = \sqrt{x^6\,x} = x^3\sqrt{x},\ x \geq 0$

Example 5 – Simplify. Let the variables be any real numbers.

a) $\sqrt{16x^2}$ b) $\sqrt{25x^2y^4}$ c) $\sqrt{x^{14}}$ d) $\sqrt{36x^2y^5}$ e) $\sqrt{32x^6y^3z^8}$

a) $4|x|$ b) $5|x|y^2$ c) $|x^7|$

d) $\sqrt{36x^2y^4\,y}$

$6|x|y^2\sqrt{y},\ y \geq 0$

e) $\sqrt{16 \cdot 2\,x^6y^2\,y\,z^8}$

$4|x^3|y\,z^4\sqrt{2y},\ y \geq 0$

Summary: For exponents on variables in even index radicals

① EVEN to ODD use abs value

② EVEN to EVEN nothing needed

③ ODD to ___, $\square \geq 0$

1.4 – Simplifying Radicals

Radicals can be written as fractional exponents, as learned in Math 10.

Examples: $\sqrt{2} = 2^{\frac{1}{2}}$ $\qquad \sqrt[3]{x} = x^{\frac{1}{3}}$ $\qquad$ Generally: $\sqrt[n]{a} = a^{\frac{1}{n}}$

Three Important Relationships of Radicals, all from Math 10:

1) $\sqrt[n]{a^n} = a,\ a \geq 0$ because $\sqrt[n]{a^n} = a^{\frac{n}{n}} = a^1$ Example: $\sqrt[2]{5^2} = 5^{\frac{2}{2}} = 5$

2) $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b},\ a, b \geq 0$ because $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = \sqrt[n]{a} \times \sqrt[n]{b}$

3) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},\ a \geq 0, b > 0$ because $\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Simplifying Expressions Containing Radicals

Example 1 – Simplify $\sqrt{20}$

$\sqrt{4 \cdot 5}$

$\sqrt{4} \cdot \sqrt{5}$

$2\sqrt{5}$

Two Methods: $\sqrt{20}$

$\sqrt{(2 \cdot 2) \cdot 5}$

$2\sqrt{5}$

20
10 2
2 5

$2 \cdot 2 \cdot 5$

It is beneficial to know the perfect squares up to 144, perfect cubes up to 125, and perfect fourths up to 81.

perfect squares: 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144

perfect cubes: 8, 27, 64, 125

perfect fourths: 16, 81

Example 2 – Simplify

a) $\sqrt{8}$

$\sqrt{4 \cdot 2}$

$\sqrt{4} \cdot \sqrt{2}$

$2\sqrt{2}$

b) $\sqrt{27}$

$\sqrt{9 \cdot 3}$

$3\sqrt{3}$

c) $3\sqrt{52}$

$3 \cdot \sqrt{4 \cdot 13}$

$3 \cdot \sqrt{4} \cdot \sqrt{13}$

$3 \cdot 2 \cdot \sqrt{13}$

$6\sqrt{13}$

d) $\sqrt[3]{24}$

$\sqrt[3]{8 \cdot 3}$

$\sqrt[3]{8} \cdot \sqrt[3]{3}$

$2\sqrt[3]{3}$

e) $5\sqrt[3]{81}$

$5\sqrt[3]{27 \cdot 3}$

$5 \cdot \sqrt[3]{27} \cdot \sqrt[3]{3}$

$5 \cdot 3 \cdot \sqrt[3]{3}$

$15\sqrt[3]{3}$

f) $\sqrt[4]{32}$

$\sqrt[4]{16 \cdot 2}$

$\sqrt[4]{16} \cdot \sqrt[4]{2}$

$2\sqrt[4]{2}$

Example 3 – Simplify (assume variables are positive)

a) $\sqrt{18x^3y^6}$

$\sqrt{9\cdot 2\, x^2 x\, y^6}$

$3xy^3\sqrt{2x}$

b) $\sqrt{63n^7p^4}$

$\sqrt{9\cdot 7\, n^6 n\, p^4}$

$3n^3p^2\sqrt{7n}$

c) $\sqrt{32x^8y^{11}}$

$\sqrt{16\cdot 2\, x^8 y^{10} y}$

$4x^4y^5\sqrt{2y}$

d) $\sqrt[3]{40a^4b^8c^{15}}$

$\sqrt[3]{8\cdot 5\, a^3 a\, b^6 b^2\, c^{15}}$

$2ab^2c^5\sqrt[3]{5ab^2}$

e) $\sqrt[3]{54a^5b^{10}}$

$\sqrt[3]{27\cdot 2\, a^3 a^2\, b^9 b}$

$3ab^3\sqrt[3]{2a^2b}$

f) $\sqrt[4]{m^7}$

$\sqrt[4]{m^4 m^3}$

$m\sqrt[4]{m^3}$

g) $\sqrt[4]{162x^3y^{11}z^5}$

$\sqrt[4]{81\cdot 2\, x^3\, y^8 y^3\, z^4 z}$

$3y^2z\sqrt[4]{2x^3y^3z}$

h) $\sqrt[3]{\frac{x^{13}}{64}}$

$\sqrt[3]{\frac{x^{12}x}{64}}$

$\frac{x^4}{4}\sqrt[3]{x}$ OR

$\frac{x^4\sqrt[3]{x}}{4}$

Changing Mixed Radicals to Entire Radicals

Example 4 – Change to Entire (assume variables are positive)

a) $4\sqrt{3}$

$\sqrt{16}\sqrt{3}$

$\sqrt{16\cdot 3}$

$\sqrt{48}$

b) $3\sqrt{5}$

$\sqrt{9}\sqrt{5}$

$\sqrt{45}$

c) $2\sqrt[3]{7}$

$\sqrt[3]{8}\sqrt[3]{7}$

$\sqrt[3]{56}$

d) $-2x\sqrt{6x}$

$-\sqrt{4x^2}\sqrt{6x}$

$-\sqrt{24x^3}$

e) $x^3\sqrt{x}$

$\sqrt{x^6}\sqrt{x}$

$\sqrt{x^7}$

f) $3a^2b\sqrt[3]{b^2c}$

$\sqrt[3]{27a^6b^3}\sqrt[3]{b^2c}$

$\sqrt[3]{27a^6b^5c}$

g) $\frac{3x^2y}{5}\sqrt[3]{2xy^2}$

$\sqrt[3]{\frac{27x^6y^3}{125}}\sqrt[3]{2xy^2}$

$\sqrt[3]{\frac{54x^7y^5}{125}}$

1.5 – Adding and Subtracting Radical Expressions

Like Radicals

'Like Radicals' work very similar to 'Like Terms'.

Simplify: $3x + 2x = 5x$

Simplify $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$

Like radicals have the same index and the same radicand

Steps for adding & subtracting like radicals:

1. Check if radicals are 'like'. Simplify any radicals you can.
2. Add or subtract coefficients, leave the 'like' radicals the same

Example 1 - Simplify

a) $7\sqrt{3} - 2\sqrt{3}$

$5\sqrt{3}$

b) $-5\sqrt[3]{10} - 6\sqrt[3]{10}$

$-11\sqrt[3]{10}$

c) $4\sqrt{2} - 5\sqrt[3]{2}$

$4\sqrt{2} - 5\sqrt[3]{2}$

d) $2\sqrt{75} + 3\sqrt{3}$

$2\sqrt{25 \cdot 3} + 3\sqrt{3}$

$10\sqrt{3} + 3\sqrt{3}$

$13\sqrt{3}$

e) $-\sqrt{27} + 3\sqrt{5} - \sqrt{80} - 2\sqrt{12}$

$-3\sqrt{3} + 3\sqrt{5} - 4\sqrt{5} - 4\sqrt{3}$

$-7\sqrt{3} - \sqrt{5}$

f) $\sqrt{9b} - 3\sqrt{16b}, \ b \geq 0$

$3\sqrt{b} - 12\sqrt{b}$

$-9\sqrt{b}$

In example f, why does b have to be greater than or equal to zero? If b is negative, radicands will be negative, so $\varnothing$

Example 2 – Simplify. Assume variables are positive values.

a) $\sqrt{27xy}+\sqrt{8xy}$

$3\sqrt{3xy}+2\sqrt{2xy}$

b) $4\sqrt[3]{16}+3\sqrt[3]{54}$

$4\sqrt[3]{8\cdot 2}+3\sqrt[3]{27\cdot 2}$

$8\sqrt[3]{2}+9\sqrt[3]{2}$

$17\sqrt[3]{2}$

c) $3x\sqrt{63y}-5\sqrt{28x^2y}$

$3x\sqrt{9\cdot 7y}-5\sqrt{4\cdot 7x^2y}$

$9x\sqrt{7y}-10x\sqrt{7y}$

$-x\sqrt{7y}$

d) $\frac{5}{2}\sqrt[3]{16x^4y^5}-xy\sqrt[3]{54xy^2}$

$\frac{5}{2}\sqrt[3]{8\cdot 2x^3xy^3y^2}-xy\sqrt[3]{27\cdot 2xy^2}$

$\frac{5}{\cancel{2}}(\cancel{2}xy)\sqrt[3]{2xy^2}-3xy\sqrt[3]{2xy^2}$

$5xy\sqrt[3]{2xy^2}-3xy\sqrt[3]{2xy^2}$

$2xy\sqrt[3]{2xy^2}$

1.6 A – Multiplying & Dividing Radical Expressions

Multiplying radicals

Example 1 - Multiply $2\sqrt{5}\,(3\sqrt{5})$

$= 2\cdot\sqrt{5}\cdot 3\cdot\sqrt{5}$

$= 2\cdot 3\cdot\sqrt{5}\cdot\sqrt{5}$

$= 6\cdot\sqrt{5\cdot 5} = 6\sqrt{25} = 6(5) = 30$

Verify your answer: $2\sqrt{5}\,(3\sqrt{5})$

$(4.472)(6.708)$

$= 30$ ✓

To multiply radicals:

① Multiply coefficients
② Multiply radicands (if index is the same)
③ Simplify.

In general: $(x\sqrt[n]{a})(y\sqrt[n]{b})$

$= xy\sqrt[n]{ab}$

where n is a natural number and x, y, a, b are real numbers. If n is even, $a \geq 0$ and $b \geq 0$

Example 2 - Simplify: a) $5\sqrt{3}\,(\sqrt{6})$

$= 5\sqrt{18}$

$= 15\sqrt{2}$

b) $2\sqrt{6}\,(4\sqrt{8})$

$= 8\sqrt{48}$

$= 32\sqrt{3}$

c) $-3\sqrt{2x}\,(4\sqrt{3x})$ $x \geq 0$

$-12\sqrt{6x^2}$

$-12x\sqrt{6}$

d) $-2\sqrt[3]{11}(4\sqrt[3]{2} - 3\sqrt[3]{3})$

$-8\sqrt[3]{22} + 6\sqrt[3]{33}$

e) $(4\sqrt{2} + 3)(\sqrt{7} - 5\sqrt{14})$ FOIL

$4\sqrt{14} - 20\sqrt{28} + 3\sqrt{7} - 15\sqrt{14}$

$4\sqrt{14} - 40\sqrt{7} + 3\sqrt{7} - 15\sqrt{14}$

$-11\sqrt{14} - 37\sqrt{7}$

f) $(\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2})$

$\sqrt[3]{x^3} - \sqrt[3]{x^2y} + \sqrt[3]{xy^2} + \sqrt[3]{x^2y} - \sqrt[3]{xy^2} + \sqrt[3]{y^3}$

$= x + y$

dividing radicals

Example 3 - Divide $\frac{6\sqrt{12}}{3\sqrt{6}}$

$= 2\sqrt{2}$

Verify your answer: $\frac{6\sqrt{12}}{3\sqrt{6}} = \frac{20.785}{7.348}$

$= 2.828$

(same as $2\sqrt{2}$)

To divide radicals:

① Divide coefficients

② Divide radicands (if index the same)

③ Simplify if possible

In general: $\frac{x\sqrt[n]{a}}{y\sqrt[n]{b}} = \frac{x}{y}\sqrt[n]{\frac{a}{b}}$ with same stipulations as multiplying radicals.

also: $y \neq 0$, $b \neq 0$ (so $b > 0$)

Example 4 - Simplify: a) $\frac{-24\sqrt[3]{14}}{8\sqrt[3]{2}}$ b) $\frac{2\sqrt{51}}{\sqrt{3}}$ c) $\frac{\sqrt{18x^3}}{\sqrt{3x}}, x > 0$

a) $-3\sqrt[3]{7}$

b) $2\sqrt{17}$

c) $\sqrt{6x^2}$

$= x\sqrt{6}$

Multiplying & Dividing Terms with Different Indices

Example 5 – Simplify $\sqrt[2]{x^3}\,(\sqrt[3]{x})$, $x \geq 0$

$= x^{\frac{3}{2}} \cdot x^{\frac{1}{3}}$ $\qquad \frac{3^{\times 3}}{2_{\times 3}} + \frac{1^{\times 2}}{3_{\times 2}} = \frac{11}{6}$

$= x^{\frac{11}{6}}$

$= \sqrt[6]{x^{11}}$

$= \sqrt[6]{x^6 x^5}$

$= x\sqrt[6]{x^5}$

Example 6 – Simplify $\frac{\sqrt[2]{x^3}}{\sqrt[3]{x}}$, $x > 0$

$x^{\frac{3}{2}} \div x^{\frac{1}{3}}$ $\qquad \frac{3^{\times 3}}{2_{\times 3}} - \frac{1^{\times 2}}{3_{\times 2}} = \frac{7}{6}$

$= x^{\frac{7}{6}}$

$= \sqrt[6]{x^7} = \sqrt[6]{x^6 x} = x\sqrt[6]{x}$

1.6 B – Rationalizing the Denominator

rationalizing the denominator

A final answer cannot have a radical in the denominator. Therefore, you may have to '**rationalize the denominator**' – a process that will eliminate the radical from the denominator without changing the value of the expression.

Example 1 - Rationalize:

If the denominator is a radical *monomial*, multiply the numerator and denominator by that radical.

a) $\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$

b) $\sqrt{\frac{2}{7}} = \frac{\sqrt{2}\cdot\sqrt{7}}{\sqrt{7}\cdot\sqrt{7}} = \frac{\sqrt{14}}{7}$

c) $\frac{6}{7\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$ $\frac{6\sqrt{5}}{7(5)} = \frac{6\sqrt{5}}{35}$

d) $\frac{2\sqrt{5}}{3\sqrt[3]{6}} \cdot \frac{(\sqrt[3]{6})^2}{(\sqrt[3]{6})^2}$ $\frac{2\sqrt{5}(\sqrt[3]{6})^2}{3(6)} = \frac{2\sqrt{5}(\sqrt[3]{6})^2}{18} = \frac{\sqrt{5}(\sqrt[3]{6})^2}{9}$

e) $\sqrt[3]{\frac{2}{y}} = \frac{\sqrt[3]{2}\cdot(\sqrt[3]{y})^2}{\sqrt[3]{y}\cdot(\sqrt[3]{y})^2} = \frac{\sqrt[3]{2y^2}}{y}$

f) $\frac{\sqrt[4]{5x}}{\sqrt[4]{10x^3}} \cdot \frac{(\sqrt[4]{10x^3})^3}{(\sqrt[4]{10x^3})^3} = \frac{\sqrt[4]{5000x^{10}}}{10x^3} = \frac{\sqrt[4]{625\cdot 8\cdot x^8\cdot x^2}}{10x^3} = \frac{5x^2\sqrt[4]{8x^2}}{10x^3} = \frac{\sqrt[4]{8x^2}}{2x}$

g) $\frac{2}{\sqrt{x+1}} \cdot \frac{\sqrt{x+1}}{\sqrt{x+1}}$ $\frac{2\sqrt{x+1}}{x+1}$

Example 2 – Rationalize:

If the denominator is a radical *binomial*, multiply the numerator & denominator by its <u>*conjugate.*</u>

conjugate of $\sqrt{x}-2$ is $\sqrt{x}+2$

a) $\frac{3}{\sqrt{x}-2} \cdot \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$\frac{3\sqrt{x}+6}{x-2\sqrt{x}+2\sqrt{x}-4} = \frac{3\sqrt{x}+6}{x-4}$

b) $\frac{2+\sqrt{2}}{3\sqrt{5}-4} \cdot \frac{(3\sqrt{5}+4)}{(3\sqrt{5}+4)}$

$\frac{6\sqrt{5}+8+3\sqrt{10}+4\sqrt{2}}{9(5)-12\sqrt{5}+12\sqrt{5}-16}$

$\frac{6\sqrt{5}+3\sqrt{10}+4\sqrt{2}+8}{45-16}$

$\frac{6\sqrt{5}+3\sqrt{10}+4\sqrt{2}+8}{29}$

c) $\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}-\sqrt{2b}}$, $a, b \geq 0$

$$\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}-\sqrt{2b}} \frac{(\sqrt{a}+\sqrt{2b})}{(\sqrt{a}+\sqrt{2b})}$$

$$= \frac{a+\sqrt{2ab}+\sqrt{2ab}+2b}{a-\sqrt{2ab}+\sqrt{2ab}-2b}$$

$$= \frac{a+2\sqrt{2ab}+2b}{a-2b}, \quad \boxed{a \neq 2b}$$

Example 3 - Simplify

a) $6\sqrt{\frac{3}{4x}}$, $x > 0$

$$= \frac{6\sqrt{3}}{\sqrt{4x}}$$

$$= \frac{6\sqrt{3}}{2\sqrt{x}}$$

$$= \frac{3\sqrt{3} \cdot \sqrt{x}}{\sqrt{x} \cdot \sqrt{x}}$$

$$= \frac{3\sqrt{3x}}{x}$$

b) $\frac{-7}{2\sqrt[3]{9p}}$, $p \neq 0$

$$\frac{-7 \cdot (\sqrt[3]{9p})^2}{2\sqrt[3]{9p} \cdot (\sqrt[3]{9p})^2}$$

$$= \frac{-7(\sqrt[3]{9p})^2}{2(9p)}$$

$$= \frac{-7(\sqrt[3]{9p})^2}{18p}$$

$$= \frac{-7(\sqrt[3]{81p^2})}{18p}$$

$$= \frac{-7(3\sqrt[3]{3p^2})}{18p}$$

$$= \frac{-21\sqrt[3]{3p^2}}{18p}$$

$$= \frac{-7\sqrt[3]{3p^2}}{6p}$$

Example 4 – The surface area of a sphere is $S = 4\pi r^2$. If the surface area of the sphere is 144 mm^2, what is the radius?

$$144 = 4\pi r^2$$

$$r^2 = \frac{144}{4\pi}$$

$$r^2 = \frac{36}{\pi}$$

$$r = \pm\sqrt{\frac{36}{\pi}}$$

reject the negative radius

$$r = \frac{\sqrt{36}}{\sqrt{\pi}}$$

$$r = \frac{6 \cdot \sqrt{\pi}}{\sqrt{\pi} \cdot \sqrt{\pi}}$$

$$r = \frac{6\sqrt{\pi}}{\pi}$$

1.7 – Radical Equations

Radical Equations are mathematical equations that include a radical, such as:

$2\sqrt{6x}-1=11$. If the index of the radical is even, there are restrictions on the variable: Since it is not possible to find the square root of a negative number, the radicand cannot be negative.

Example 1 – Find the restriction on the variable:

a) $2\sqrt{6x}-1=11$

$\frac{6x}{6} \geq \frac{0}{6}$

$\boxed{x \geq 0}$

b) $\sqrt{x+2}=49$

$x+2 \geq 0$

$\boxed{x \geq -2}$

c) $7\sqrt{-2x+3}=35$

$-2x+3 \geq 0$

$\frac{-2x}{-2} \geq \frac{-3}{-2}$ flip

$\boxed{x \leq \frac{3}{2}}$

d) $\sqrt{3x+4}=\sqrt{2x-4}$

$3x+4 \geq 0$

$3x \geq -4$

$x \geq \frac{-4}{3}$

$2x-4 \geq 0$

$2x \geq 4$

$x \geq 2$

So

$\boxed{x \geq 2}$

Steps to solving radical equations:

1. Find the restrictions on the variable in the radicand (if the index is even). Remember, the radicand must be set to ≥ 0 and then solved (if you multiply or divide by a negative number to both sides, FLIP the inequality).
2. Get the radical all by itself on one side of the equation.
3. If the index is 2, square both sides (if index is 3, cube both sides, etc.) and then solve for the variable.
4. See if the solution if affected by the restriction.
5. Check the answer using the original equation to see if solutions are valid or extraneous.

Example 2 – Solve a) $2\sqrt{6x}-1=11$ $\boxed{x \geq 0}$

$2\sqrt{6x}-1=11$

$+1 \quad +1$

$\frac{\cancel{2}\sqrt{6x}}{\cancel{2}}=\frac{12}{2}$

$\sqrt{6x}=6$

$(\sqrt{6x})^2=6^2$

$6x=36$

$x=6$

Check

LS	RS
$2\sqrt{6x}-1$	11
$2\sqrt{6(6)}-1$	
$2\sqrt{36}-1$	
$2(6)-1$	
$12-1$	
11	

✓

b) $-8+\sqrt{\frac{3y}{5}}=2$ $\boxed{y \geq 0}$

$\cancel{-8}+\sqrt{\frac{3y}{5}}=2$

$\cancel{+8} \quad +8$

$\sqrt{\frac{3y}{5}}=10$

$\left(\sqrt{\frac{3y}{5}}\right)^2=10^2$

$\frac{3y}{5}=100$

$3y=500$

$y=\frac{500}{3}$

Check

LS	RS
$-8+\sqrt{\frac{3y}{5}}$	2
$-8+\sqrt{\frac{3(\frac{500}{3})}{5}}$	
$-8+\sqrt{100}$	
$-8+10$	
2	

✓

Example 3 – Solve a) $4+\sqrt{2x-3}=1$ $\boxed{x \geq \frac{3}{2}}$ b) $-2\sqrt{x-5}=16$ $\boxed{x \geq 5}$

$2x-3 \geq 0$
$2x \geq 3$
$x \geq \frac{3}{2}$

$4+\sqrt{2x-3}=1$
$-4 \quad -4$
$(\sqrt{2x-3})^2=(-3)^2$
$2x-3=9$
$2x=12$
$\cancel{x=6}$
extraneous
No solutions $\varnothing$

check

LS	RS
$4+\sqrt{2(6)-3}$	1
$4+\sqrt{9}$	
$4+3$	
7	X

$\sqrt{x-5}=-8$
NO SOLUTIONS

* If there is just a radical on one side and just a negative constant on the other, there are NO SOLUTIONS

* If two radicals, get one on each side, then square!

Example 4 - Solve

a) $\sqrt{10x-7}=3\sqrt{x}$ $\boxed{x \geq \frac{7}{10}}$ b) $2\sqrt{x}=\sqrt{7x+6}$ $\boxed{x \geq 0}$

$10x-7 \geq 0$
$10x \geq 7$
$x \geq \frac{7}{10}$
and
$x \geq 0$
so
$\boxed{x \geq \frac{7}{10}}$

$(\sqrt{10x-7})^2=(3\sqrt{x})^2$
$10x-7=9x$
$x=7$

check

RS	LS
$\sqrt{10(7)-7}$	$3\sqrt{7}$
$\sqrt{63}$	
$3\sqrt{7}$	✓

$x \geq 0$
and
$7x+6 \geq 0$
$7x \geq -6$
$x \geq \frac{-6}{7}$
so
$x \geq 0$

$(2\sqrt{x})^2=(\sqrt{7x+6})^2$
$4x=7x+6$
$-3x=6$
$x=-2$
does not fulfill restric!
So no solutions!

Example 5 – Solve

a) $\sqrt{x+1}=x-1$ $\boxed{x \geq -1}$ b) $m-\sqrt{2m+3}=6$ $\boxed{m \geq -\frac{3}{2}}$

$(\sqrt{x+1})^2=(x-1)^2$
$x+1=(x-1)(x-1)$
$x+1=x^2-x-x+1$
$x+1=x^2-2x+1$
$x^2-3x=0$
$x(x-3)=0$
ext $x=\cancel{0},3$ $x=3$

$x=0$

LS	RS
$\sqrt{x+1}$	$x-1$
$\sqrt{0+1}$	$0-1$
$\sqrt{1}$	-1
1	X

$x=3$

LS	RS
$\sqrt{3+1}$	$3-1$
$\sqrt{4}$	2
2	✓

$2m+3 \geq 0$
$2m \geq -3$
$m \geq \frac{-3}{2}$

$m-6=\sqrt{2m+3}$
$(m-6)^2=(\sqrt{2m+3})^2$
$(m-6)(m-6)=2m+3$
$m^2-12m+36=2m+3$
$m^2-14m+33=0$
$(m-11)(m-3)=0$
$m=11$, $\cancel{3}$ ext
$M=11$

$m=11$

LS	RS
$m-\sqrt{2m+3}$	6
$11-\sqrt{2(11)+3}$	
$11-\sqrt{25}$	
$11-5$	
6	✓

$\cancel{m=3}$

LS	RS
$3-\sqrt{2(3)+3}$	6
$3-\sqrt{6+3}$	
$3-\sqrt{9}$	
$3-3$	
0	X