Arithmetic Sequences Notes Key

Focus: To understand and apply the concept of arithmetic sequences

Warmup - The starting salary of an employee is $21 250. If a raise of $1250 is given each year, in how many years will the employee's salary be $50 000?

1	2	3	4, ...,	n
21250	22500	23750	25000	50000

There will be 23 raises (no raise in 1^{st} year)

Let x = number of raises

$21250 + 1250x = 50\,000$

$x = 23$

so 24 years to get to $50 000

The problem above can be solved using a formula, which we will develop:

vocabulary

A **sequence** is simply an ordered list of numbers (called **terms**) that follow a pattern so that the next term can be determined.

Example: *4, 7, 10, 13, 16* $t_1 =$ 4 $n =$ 5 $t_4 =$ 13 $d =$ 3

- The **first term** in the sequence is labeled t_1
- The **number of terms** in the sequence is n
- Any term of the sequence is t_n (read *t* sub *n*), dependent on the value of n. For example, the **third term** is t_3, the **eighth term** is t_8 etc.
- A **finite sequence** has a finite number of terms whereas an **infinite sequence** has an infinite number of terms
- An **ARITHMETIC SEQUENCE** is an ordered list of terms in which the difference between consecutive terms is constant (a **common difference** (d))

For the salary problem, what is t_1? What is d?

$t_1 =$ 21 250 $d =$ 1250

How can you find t_2 using t_1 and d? $t_2 = 21250 + 1250$ in general: $t_2 = t_1 + 1d$

How can you find t_3 using t_1 and d? $t_3 = 21250 + 2(1250)$ in general: $t_3 = t_1 + 2d$

How can you find t_4 using t_1 and d? $t_4 = 21250 + 3(1250)$ in general: $t_4 = t_1 + 3d$

Can you develop a formula for the general term of an arithmetic sequence?

formula for an arithmetic sequence

$$\boxed{t_n = t_1 + (n-1)d}$$

Use the formula to answer the problem from the top of the page:

$t_1 = 21\,250$

$d = 1250$

$t_n = 50\,000$

Find n

$t_n = t_1 + (n-1)d$

$50\,000 = 21250 + (n-1)1250$

$50\,000 = \cancel{21250} + (n-1)1250$

$-21250 \quad -\cancel{21250}$

$\frac{28\,750}{1250} = \frac{\cancel{1250}(n-1)}{\cancel{1250}}$

$23 = n - 1$

$n = 24$

Example 1 – Find the 33[rd] term of the arithmetic sequence: *-10, -4, 2, 8, 14, ...*

$t_1 = -10$
$d = 6$
$n = 33$

$t_n = t_1 + (n-1)d$
$t_{33} = -10 + (33-1)(6)$
$t_{33} = -10 + 32(6)$
$t_{33} = -10 + 192$

$t_{33} = 182$

Example 2 – You notice a few carpenter ants in your basement and by the end of the first month there are 40 ants. The growth produces an arithmetic sequence in which the number of ants increases by approximately 80 ants each month. How many months in total until the colony reaches a population of 3000 ants?

$t_1 = 40$
$d = 80$
$t_n = 3000$
$n =$ months

$t_n = t_1 + (n-1)d$
$3000 = 40 + (n-1)(80)$
$\frac{2960}{80} = \frac{(n-1)(80)}{80}$
$37 = n - 1$
$n = 38$

It will take 38 months

Example 3 – A furnace technician charges \$35 for making a house call, plus \$46 per hour.

a) Generate the possible charges for the first 4 hours of time.
b) What is the charge for 10 hours of time?

$t_1 = 35 + 46 = 81$
$t_2 = 81 + 46 = 127$
$t_3 = 81 + 2(46) = 173$
$t_4 = 81 + 3(46) = 219$

a) 81, 127, 173, 219

b) $t_1 = 81$
$d = 46$
$n = 10$

$t_{10} = 81 + (10-1)(46)$
$t_{10} = 81 + (9)(46)$
$t_{10} = \$495$

Example 4 – If an arithmetic sequence starts with -21, and the 15[th] term is 105, find the common difference, *d*.

$t_1 = -21$
$n = 15$
$t_{15} = 105$

$t_n = t_1 + (n-1)d$
$105 = -21 + (15-1)d$
$105 = -21 + 14d$
$126 = 14d$
$d = 9$

Arithmetic Series

Focus: To understand and apply the concept of Arithmetic Series.

Warmup – When the famous mathematician Gauss was young, his teacher tried to keep him busy and asked him to sum the numbers from 1 to 100. It only took Gauss a few minutes! How did he doe this so fast?

Gauss multiplied 101 by 100 and divided by 2: $\frac{(101)(100)}{2} = 5050$

Gauss' method

Use Gauss' Method to calculate the sum of the numbers from 1 to 8 (show your work).

$$\begin{array}{l} 1+2+3+4+5+6+7+8 \\ 8+7+6+5+4+3+2+1 \\ \hline 9+9+9+9+9+9+9+9 \end{array} \qquad \frac{9(8)}{2} = 36$$

Building off of the example above, we'll derive a general formula for the sum of an arithmetic series using S_n as the sum of an arithmetic series, t_1 as the first term, n as the number of terms, and d as the common difference.

What does the 100 represent in Gauss' formula? Number of terms, n

What does the 101 represent? $t_1 + t_n$ first term plus last term

Write a general formula for Gauss' method:

$$S_n = \frac{n(t_1+t_n)}{2} = \frac{n}{2}(t_1+t_n)$$

formula for the sum of an arithmetic series

The sum of an arithmetic series can be determined using the formula

$$S_n = \frac{n}{2}(t_1+t_n)$$

where t_1 = first term, t_n is the last term, n is the number of terms, and S_n is the sum of the first n terms

From last class, $t_n = t_1 + (n-1)d$, so the formula for S_n can be written another way.

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$$S_n = \frac{n}{2}(t_1 + (t_1 + (n-1)d))$$

$$S_n = \frac{n}{2}[2t_1 + (n-1)d]$$

The sum of an arithmetic series can also be determined using the formula:

$$S_n = \frac{n}{2}[2t_1 + (n-1)d]$$

Example 1 – Find the sum of: -12 + -5 + 2 + 9 + 16 + 23 + 30 + 37 + 44 + 51

$t_1 = -12$

$d = 7$

$n = 10$

$t_{10} = 51$

$S_n = \frac{n}{2}(t_1+t_n)$

$S_{10} = \frac{10}{2}(-12+51)$

$S_{10} = 5(39) = 195$

$\boxed{OR}$ $S_n = \frac{n}{2}[2t_1 + (n-1)d]$

$S_{10} = \frac{10}{2}[2(-12) + (10-1)(7)]$

$S_{10} = 5[-24+63] = 195$

Example 2 – Fireflies flash in patterns to signal location or ward off predators. Suppose a firely flashes twice in the first minute, four times in the second minute, and six times in the third minute.

a) If this pattern continues, what is the number of flashes **in the** 42^{nd} minute?
b) What is the **total number** of flashes for the male firefly after 42 minutes?

a) Find t_{42}
- use formula from last day

$t_1 = 2$
$d = 2$
$n = 42$

$t_{42} = 2 + (42-1)(2)$
$t_{42} = 2 + 82$
$t_{42} = 84$ flashes in the 42^{nd} minute

b) Find S_{42}

$t_1 = 2$
$d = 2$
$n = 42$
$t_n = 84$

$S_n = \frac{n}{2}[t_1 + t_n]$
$S_{42} = \frac{42}{2}[2 + 84]$
$S_{42} = 21(86) = \underline{1806 \text{ flashes}}$

using the correct formula

Example 3 – For the arithmetic series, determine the value of n: $t_1 = -6$, $t_n = 21$, $S_n = 75$. Then, for the following series, determine t_1: $d = 0.5$, $S_n = 218.5$, $n = 23$.

$t_1 = -6$
$t_n = 21$
$S_n = 75$
$n = ?$

$S_n = \frac{n}{2}(t_1 + t_n)$
$75 = \frac{n}{2}(-6 + 21)$
$75 = \frac{n}{2}(15)$
$75 = 7.5n$
$n = 10$

$d = 0.5$
$S_n = 218.5$
$n = 23$
$t_1 = ?$

$S_n = \frac{n}{2}[2t_1 + (n-1)d]$
$218.5 = \frac{23}{2}[2t_1 + (23-1)(0.5)]$
$218.5 = 11.5[2t_1 + 11]$
$19 = 2t_1 + 11$
$8 = 2t_1$
$t_1 = 4$

Sigma Notation : greek letter sigma, Σ, means 'sum' in math

$$\sum_{n=\_}^{\_} \boxed{}$$

last value of n
formula for each term
first value of n

Example:

Last value of n

$$\sum_{n=1}^{6} 2n$$

Formula for each term

First value of n

$$2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6)$$

Example 4 – Find (a) $\sum_{n=3}^{7} 2n + 1$ (b) $\sum_{n=1}^{6} 3 + n$

(a)

$[2(3)+1] + [2(4)+1] + [2(5)+1] + [2(6)+1] + [2(7)+1]$
$7 + 9 + 11 + 13 + 15$
$S_n = \frac{n}{2}(t_1 + t_n)$
$S_5 = \frac{5}{2}(7 + 15) = \underline{55}$

(b)

$[3+1] + [3+2] + [3+3] + [3+4] + [3+5] + [3+6]$
$4 + 5 + 6 + 7 + 8 + 9$
$S_6 = \frac{6}{2}(4 + 9) = \underline{39}$

Simple Interest

Focus: To understand and apply the simple interest formula to situations.

Warmup:

1) Change 62% to dec
2) Change 8% to dec
3) Change 3.5% to dec
4) Change 0.09 to %

1) $62\% \div 100 = 0.62$ OR $62. = 0.62$
2) $8 = 0.08$
3) 0.035
4) $0.09 = 9\%$

If you get a loan, you usually have to pay it back with interest.
a) What does this mean?

b) Why do you think this is the case?

a) You have to pay the loan back plus extra money

b) There is no advantage to loan money if this wasn't the case

Can you define **Interest?**

INTEREST: money paid at a particular rate for the use of money lent, or for delaying the repayment of a debt.

Sometimes, interest works against you. Describe a scenario where this is the case.

You borrow money, and then have to pay it back with interest

Sometimes, interest works for you. Describe a scenario where this is the case.

You invest your money or put it into an account, and you make interest on your money

Why does a bank store money for you and pay you interest, yet they make huge profits?

they use your money and lend it out to others, who pay it back with interest, and you get a small share of that interest

Ex1 – Suppose you invested \$100 at 4% interest for a year. How much would you now have?

\$100 invested = PRINCIPAL (starting value)

4% = interest rate = 0.04 in decimal

1 year $\quad \$100 \times 0.04 \times 1 = \$4$

You now have $\$100 + \$4 = \$104$

Ex2 – (a) Suppose you invested $42 000 at 6% for a year. How much would you now have?

$42000 \times 0.06 \times 1 = 2520$

$42000 + 2520 = \$44520$

b) What if you invested it at simple interest for 12 years? How much would you have?

$42000 \times 0.06 \times 12 = \$30240$

$(\text{OR } 2520 \times 12) = 30240$

$42000 + 30240 = 72240$

Here is the formula for **Simple Interest:**

$$\boxed{\begin{array}{l} I = Prt \\ I = \text{interest } \$ \qquad t = \text{time (in years)} \\ P = \text{principal } \$ \\ r = \text{interest rate} \end{array}}$$

Ex3 – Use the formula to find out how much interest you would pay if you borrowed $12 000 at 7.5% interest over 8 years.

$P = \$12000$ $\quad I = Prt$

$r = 0.075$ $\quad I = (12000)(0.075)(8)$

$t = 8$ $\quad I = \$7200$

You would pay $7200 in interest

You would owe back $12000 + 7200 = \$19200$

Sometimes, the situation is under 1 year. In that case, make the applicable **proper fraction** for ***t***.

Ex4 – You borrow $5000 for 7 months at 12% simple interest. How much will you pay back in total?

12 months in a year

$t = \frac{7}{12}$ (a little over half a year)

$P = 5000$

$r = 0.12$

$I = Prt$

$I = (5000)(0.12)(\frac{7}{12})$

$I = \$350$

Total Payback $= 5000 + 350 = \$5350$

Ex5 – Suppose you won the lottery and invested $3 000 000 at simple interest at 5%. How much interest would you every week?

$P = 3000000$

$r = 0.05$

52 wks per year $\quad t = \frac{1}{52}$

$I = Prt$

$I = (3000000)(0.05)(\frac{1}{52})$

$I = \$2884.62$

Ex6 – You make $540 interest on $4000 over 3 years. What is the interest percentage?

$I = 540$

$P = 4000$

$t = 3$

$I = Prt$

$540 = (4000)(r)(3)$

$\frac{540}{12000} = \frac{12000r}{12000}$

$0.045 = r$

4.5%

Compound Interest

Focus: To understand and apply the compound interest formula to situations.

Warmup

Evaluate:

a) 2^7

b) 4^8

c) $(1 + 0.075)^{30}$

d) $(1 + 0.12)^4$

a) $2^7 = 128$ b) $4^8 = 65536$

c) $(1+0.075)^{30} = (1.075)^{30} = 8.755$

d) $(1+0.12)^4 = (1.12)^4 = 1.5735$

Use the exponent button on your calculator: x^y or y^x or ^ or $x^{\square}$

From yesterday, simple interest is when you pay/earn interest on the principal (starting) value:

P = \$5000, r = 0.10 (10%)

Ex: Year 1: 10% of \$5000 is \$500 so you make \$500 interest

Year 2: 10% of \$5000 is \$500 so you make \$500 interest

etc...etc...so you make \$500 interest each year

Can you describe what <u>compound</u> interest is?

when you earn interest on the interest as well as the principal!

Ex: Year 1: 10% of \$5000 is \$500 so you make \$500 interest

Year 2: now have 5500; 10% of 5500 is \$550 interest

Year 3: now have \$6050; 10% of 6050 is \$605 interest

Year 4: now have 6655 etc...

This table shows the clear difference between simple and compound interest:

Year	Simple Interest			Compound Interest		
	P	I	Total	P	I	Total
1	5000	500	5500	5000	500	5500
2	5000	500	6000	5500	550	6050
3	5000	500	6500	6050	605	6655
4	5000	500	7000	6655	665.50	7320.50
5	5000	500	7500	7320.50	732.05	8052.55
6	5000	500	8000	8052.55	805.26	8857.81
7	5000	500	8500	8857.81	885.78	9743.59

**Remember: for compound interest, the interest added each year becomes part of next year's principal, so you make interest on the interest!!*

Ex1 – Eva invests \$30 000 for 3 years at 5% cmpd int. How much will she have?

Year 1: (30 000)(0.05)(1) = \$1500   30 000 + 1500 = 31 500

Year 2: (31 500)(0.05)(1) = 1575   31 500 + 1575 = 33 075

yr 3: (33075)(0.05)(1) = 1653.75

\$33 075 + 1653.75 = \$34 728.75

Ex1 –continued
Or, you can find the answer using the compound interest formula.

$$A = P(1+r)^t$$

A = total amount
P = principal
r = interest rate
t = years

Using the formula:

$A = P(1+r)^t$
$A = 30000(1+0.05)^3$
$= 30000(1.05)^3$
$= \$34728.75$

Ex2 – Watch **'Act 1'** of ***Fry's Bank***.
What do we know so far?

$P = \$0.93$
2.25%, $r = 0.0225$
$t = 1000$

Ex3 – Jesse invests \$3500 at 4% cmpd int over 9 years. How much will he have?

$A = P(1+r)^t$
$P = 3500$
$r = 0.04$
$t = 9$

$A = 3500(1+0.04)^9$
$A = 3500(1.04)^9$
$A = \$4981.59$

Ex4 – Fran invests \$40 000 at 4.75% over 14 years. How much more will he have if he uses compound interest rather than simple?

$P = 40\,000$
$r = 0.0475$
$t = 14$

Cmpd:
$A = P(1+r)^t$
$A = 40000(1+0.0475)^{14}$
$A = 40000(1.0475)^{14}$
$A = \$76\,597.82$

Simple:
$I = Prt$
$I = (40000)(0.0475)(14)$
$I = 26\,600$
Total: $40\,000 + 26\,600$
$= 66\,600$

difference $= 76597.82 - 66\,600.00$
$\$9997.82$

Ex5 – Maddy has \$6303.07 after investing her principal for 6 years at 7% compound int. How much did she start with?

$A = 6303.07$
$t = 6$
$r = 0.07$

$A = P(1+r)^t$
$6303.07 = P(1+0.07)^6$

$6303.07 = P(1.07)^6$
$\frac{6303.07}{1.5} = \frac{P(1.5)}{1.5}$
$\$4202.05 = P$

Ex6 – Let's work out ***Fry's Bank*** & then watch the video solution.

$A = P(1+r)^t$
$A = 0.93(1+0.0225)^{1000}$
$A = 0.93(1.0225)^{1000}$
$A = \$4\,283\,508\,451$ ← Wow!!

How much would Fry have if it was simple interest instead?
$I = Prt$
$I = (0.93)(0.0225)(1000)$
$I = \$20.93$
total $= 0.93 + 20.93 = \$21.86$

Let's explore the **Compound Interest Formula** a bit further to see why it works:

$A = P(1+r)^t$

Why $(1+r)$?

Say $P = 1000$
$r = 10\% = 0.10$

$P \times r = 1000 \times 0.10 = 100$ ↑ just interest

$P \times (1+r) = 1000(1+0.10)$
$= 1000(1.1) = 1100$ ← Principal plus interest

$P = 1000$
$r = 0.10$
$t = 3$

$A = 1000(1+0.10)^3$ ← why here?
$A = 1000(1.1)(1.1)(1.1)$
$= 1100(1.1)(1.1)$ After yr 1
$= 1210(1.1)$ After yr 2
$= 1331$ after yr 3